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A TREATISE

ON

TRIGONOMETRY,

PLANE AND SPHERICAL,

WITH ITS APPLICATION TO

NAVIGATION AND SURVEYING, NAUTICAL AND PRACTICAL
ASTRONOMY AND GEODESY,

WITH

LOGARITHMIC, TRIGONOMETRICAL, AND NAUTICAL

TABLES.

FOR THE USE OF SCHOOLS AND COLLEGES.

A NEW EDITION, WITH EXTENSIVE ADDITIONS AND IMPROVEMENTS.

BY THE
REV. CHARLES W. HACKLEY, S.T.D.,
Prof. of Mathematics and Astronomy in Columbia College.

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PART OF

PREFACE TO THE FIRST EDITION.

Analytical Trigonometry has always been to the majority of students a dry and difficult study. A conviction that it might be rendered easy and interesting to all who have a tolerable acquaintance with Algebra and Geometry, has led to the production of the present work. The faults of former treatises on this subject, which have detracted from their usefulness as books of instruction, appear to be these:

1. A too sudden transition from Geometry to Trigonometry, in consequence of which, the first efforts of the learner are in the dark as to the object of his pursuit.

2. A tedious succession of general formulas at the commencement, the use and application of which are so long delayed as to produce weariness and discouragement before there is any apparent fruit to reward labor.

3. Too much abridgment in the demonstration, and particularly in the derivation of the algebraic results.

The author is aware of the importance attached to the exercise of intellect required to discover the connexion between propositions whose mutual dependence is shown by intermediate links which the mind must supply unaided, but it will be admitted on the other hand, that the ordinary term of study is too limited, and the field of knowledge in this department too extensive, to afford the loss of time which such a mode occasions. Besides, there will be abundant scope for this kind of exercise, in a more matured familiarity with mathematical reasoning, for which the shortening of labor here, will leave additional room.

PREFACE TO THE NEW EDITION.

THE old edition has been entirely remodelled, and vast additions of valuable matter which have been some years in collecting, or are the results of recent improvements in science, have been made. The present work begins with some constructions of triangles according to the rules given in geometry, followed by others in which scales of equal parts and protractors are employed, showing at once and distinctly, what is to be understood by the solution of a triangle, and the value of trigonometry in the measurement of inaccessible heights and distances.

The evident inaccuracy in the use of instruments leads the learner to perceive the necessity of a more exact and certain method, and prepares him to enter with satisfaction upon the study of Analytical Trigonometry.

The explanation of the Trigonometrical Lines has been prepared with great care, and it is believed that considerable improvement in the method of exhibiting their changes will be observed. Their application to the solution of triangles is immediately shown in a few cases, with the help of a table of natural sines and cosines at the end.

Then follows a full exposition of the theory and use of logarithms, with every variety of example, including an explanation of the Tables at the end. The use is also taught of the tables of Callet, the tables in highest repute, an American edition of which is known as Hassler's tables.*

Part I. concludes with the application of logarithms and logarithmic sines, tangents, &c., to a number of practical examples in heights and distances, involving every case in the solution of plane triangles.

After this a few pages of miscellaneous exercises occur, which, with those in fir
print scattered through the 1st Part, will serve to give greater skill to the better cla
of students. Appendix I., which follows next in order, contains a vast variety
general formulas, succinct methods of solution, and methods advantageous in p
ticular cases, methods of treating small arcs, resolutions of Algebraic equation
the aid of Trigonometry, various expressions for the area of a triangle in terms o
angles and sides, effects of errors of observation on results; in short, everyt
necessary for a comprehensive knowledge of Trigonometry.†

---

* The German tables most in use are those of Vega and those of Köhler.

† Navigation and Surveying, if to be studied, should be taken up immediatel
Plane Trigonometry.

Part II. contains Spherical Trigonometry. Particular care has been taken to render the demonstrations here, plain and easy, and to avoid all unnecessary repetition and complication.

It was found that the introduction of a few celestial circles, such for the most part as the study of geography may be supposed to have already rendered familiar, would afford an opportunity for making all the examples of Spherical Trigonometry Astronomical.

The use of the hour angle and of different kinds of time has led also to the introduction of a full description of the transit instrument and its various adjustments, the theory of which depends on Spherical Trigonometry, and is given in all its details.

The practical character of the problems is a peculiar feature in the plan of the present work. The consideration which led to it was that since Trigonometry had grown out of the actual wants of men in these very particulars, if they were sufficiently interesting to stimulate discovery, they would also incite to the study of what is already known. The analytic method, though not always practicable before the mind is somewhat furnished, is doubtless by far the best method of training. Besides this general reason for introducing Astronomical problems here, it was deemed useful thus to prepare the way for the study of Astronomy whilst the formulas and rules of Trigonometry were fresh in the memory, and to prevent that neglect of the Trigonometrical Solutions of Astronomy, which is apt to result from the trouble of recalling what has been long laid aside. It was thought, too, that this foretaste of Astronomy might excite a relish for that study.

The examination questions will be found convenient for students preparing for examination on Trigonometry, or for those studying without the aid of a teacher.

Appendix II., which follows Spherical Trigonometry, is of a character analogous to Appendix I.

Part III. exhibits a pleasing and useful application of Plane Trigonometry to the principles of navigation. This will be found a very complete treatise on the subject in small compass. The appendix to this part, App. III., includes great circle sailing, a method not usually treated in works on navigation, nor much used at present at sea; but as it serves to shorten voyages, and has no practical inconveniences in the case of steamers, which class of vessels is becoming numerous on the ocean, it cannot longer with propriety be omitted. Sumner's method is also here introduced.

Part IV. is a very complete treatise of surveying, which, by reducing the subject rigorously to its essential elements, is brought within a small space. Besides what is contained in ordinary treatises, including a full description of all the surveying instruments, will be found the methods of surveying railroads and canals, the principles of Topography, and a new method of Hydrographic Surveying.

Part V., which treats of Nautical and Practical Astronomy, contains a complete description of all the Astronomical instruments used at sea and in observatories, a thorough investigation of the theory of their adjustments, and of the corrections to be applied to the observations for errors of adjustment, the use of nearly every part of the nautical almanac and tables of corrections for determining the co-ordinates of the true places of the heavenly bodies, and the solutions in Spherical Trigonometry necessary for converting one set of co-ordinates into another, with all the best methods of determining latitude and longitude, either on land or at sea. App. V. contains the description of the reflecting circle and mural circle, the determination of latitude by circummeridian altitudes, by the method of Littrow, and by an altitude of *the pole star* out of the meridian.

Part VI. contains the necessary instruction for conducting a geodetic survey on a scale of sufficient magnitude to require not merely the spherical figure, but also the spheroidal figure of the earth to be taken into consideration. When the formulas in this part involve the theory of conic sections, they are given and the use taught, but the demonstrations are reserved for the last appendix, in which the calculus is freely introduced when necessary. The subject commences with the modes of measuring bases, with an account of the beautiful improvements in the base apparatus recently made in this country, and the formulas of reduction to the level of the neighboring seas. Then follows a description of the great theodolite, and the methods of conducting the observations of the great or primary triangulation, the modes of verifying and correcting the observed spherical angles, and of computing the elements of the spherical triangles. Then the methods of determining geodetically the differences of latitude, longitude, and azimuth of the stations at the vertices of the triangles, with the construction of maps and the explanation of the necessary tables. Then the best methods of conducting the Astronomical observations for latitude, longitude, and azimuths. The description of the instruments and modes of conducting the magnetic observations, and the use of the formulas for determining the elements of terrestrial magnetism.

App. VI. describes the equatorial, the altitude and azimuth instrument, the prime vertical transit, and gives theorems for determining the size and figure of the earth, &c.

The methods given in this geodetic treatise are those employed upon the coast survey of the United States.*

The tables include a table of logarithms, of numbers, of logarithmic sines, tangents, cosines, cotangents, secants and cosecants;† a table of natural sines and cosines, a table of difference of latitude and departure for every point and quarter point of the quadrant, a table of Rhumbs, a table of meridional parts, Workman's table for the correction of the middle latitude, a table of refractions, with corrections for the states of the barometer and thermometer, a table for dip or depression of the horizon, a table of the sun's parallax in altitude, of the contraction of the sun's or moon's vertical semi-diameter from refraction, of the augmentation of the moon's semi-diameter with its altitude, a table of proportional logarithms, a table of the reductions of the moon's equatorial parallax for the spheroidal figure of the earth, and finally a table of natural versed sines for reducing observations to the meridian.

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* These are in some respects superior to the latest and best European methods. The author has to acknowledge the politeness of the accomplished superintendent of the coast survey in furnishing every facility for obtaining information.

† The last two are not usually found in the best tables. The method of taking out the difference for the seconds in these tables is new and expeditious.

Besides these, other small tables and specimens of tables are scattered throughout the work.

Most of the tables are printed from the beautiful and accurate stereotype plates of the tables accompanying Bowditch's Navigator, by permission of the proprietor, Mr. G. W. Blunt.

The author has to acknowledge the kindness of Prof. CHAUVENET, of the U. S. Naval Academy, in permitting the use of his valuable paper on Unlimited Spherical Triangles, first introduced by Gauss. It will be found in Appendix II., as contained in the Astronomical Journal, with some slight modifications and explanatory notes.

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PART I.

PLANE TRIGONOMETRY.

1. THE term TRIGONOMETRY is compounded of two Greek words, τριγωνος, a triangle, and μετρον, measure, signifying literally the measurement of triangles. It has for its object to determine the unknown parts of a triangle when a sufficient number of the parts is known.

By parts or elements of a triangle are understood commonly the sides and angles, though trigonometry properly includes the measurement of the surface also.

There will accordingly be six elements of every triangle, namely the three sides and the three angles.

2. It has been proved (Plane Geom., Theorems 1, 2, and 5), that when two triangles have three elements, one of which is a side, in the one, equal respectively to the corresponding elements in the other, the triangles are identical.

One element must be a side, because if the three angles only were equal respectively in the two triangles they would be but similar (Plane Geom., Theorem 63); that is, alike in shape but not necessarily in size.

Since all triangles which have three elements equal, are by consequence equal, it is said that three given elements determine a triangle; that is, with these three given elements, but one triangle can be formed.

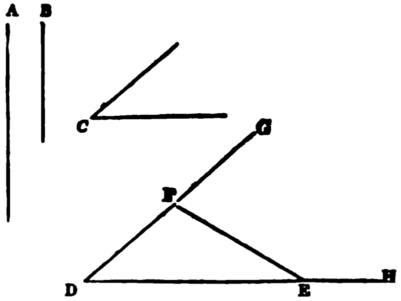
There is one exception to this principle, pointed out in Prob. 8, Plane Geom., where two sides and the angle opposite one of them are given, in which case two triangles can be constructed with the given elements.

3. Three elements of a plane triangle being given then (except they be the three angles), it ought to be possible to find the other three, since these are fixed by their dependence upon the three given.

This may be accomplished with sufficient accuracy for many purposes, by means of constructions such as are exhibited at Problems 5 and 8 of Plane Geometry.

We shall repeat one of these constructions, enunciating the problem somewhat differently.

The two sides and included angle of a triangle being given, let it be required to find the remaining side and the other two angles.

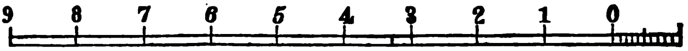


Let A and B be the two given sides, and C the given included angle. Draw two lines DH and DG of indefinite length, making with each other an angle equal to the given angle C. Lay off on the first of these the given line A from D to E, and on the second the given line B from D to F. Join EF. The only possible triangle DEF will thus be formed with the three given elements, in which EF will be the required side, and E and F the required angles.

The finding the unknown elements of a triangle by means of those which are given is called its solution.

4 The method of solution just exhibited is rendered more practically useful by the employment of scales of equal parts and protractors.

The most simple form of the scale of equal parts is shown in the annexed figure.

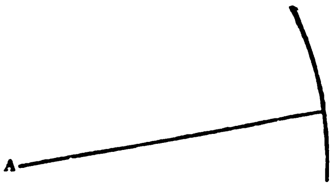


It is a straight rule divided into any number of equal parts; in this example ten, and one of these again into ten, so that the smallest division is one hundredth of the whole length of the rule.

The following is the manner of using it.

Suppose that it is required to draw upon paper a line equal in length to 56.

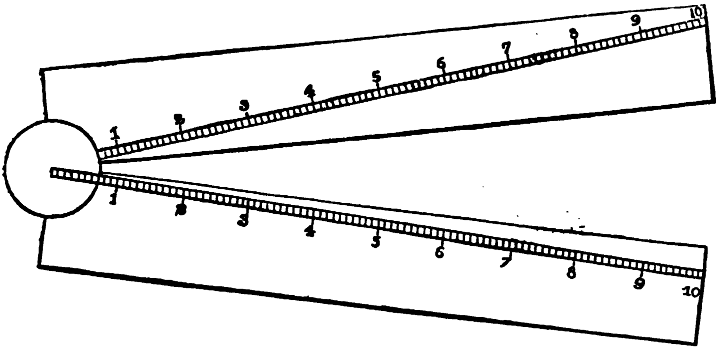
Place one foot of a pair of dividers at the line of division marked 5, and extend them till the other foot reaches exactly to the sixth smaller division mark on the right of 0; the feet of the dividers will then be at a distance of 56 apart. To draw now the required line upon paper, let A be the point from which it is to be drawn. Placing one foot of the dividers at A, extended the



distance 56 obtained from the scale, describe with the other an arc of a circle on the side towards which the line is to be drawn; then from A draw the line in the proper direction, terminating it at the arc before described, and it will be the line required.

Another line of 42 being measured from the scale and laid down upon the paper, the two lines will be in the ratio of 56 to 42. If they are lines upon a map, and the first corresponds to a line of 56 feet upon the ground, the second will correspond to a line of 42 feet. If the first represent 56 yards, or chains, or miles, the second will represent 42 yards, or chains, or miles. And in general lines upon the same drawing which are measured in parts of the same scale must be understood to be expressed in units of the same kind.

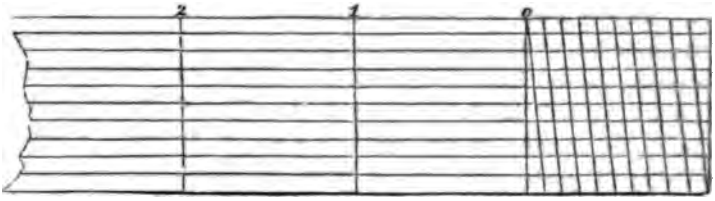
The *sectoral* scale of equal parts consists of a ruler of two arms moving on a hinge, each arm being divided into a number (usually 100) of equal parts. To set this scale to any size, say 40 parts to the inch, the arms must be separated by turning them round the hinge, till a pair of dividers opened to the distance of an inch will extend exactly from the division marked 40 on one arm, to that marked 40 on the other.



If now any other distance be required upon a scale of 40 parts to the inch, as for instance the distance 65, the dividers must be opened till they will extend from the 65th division on the one arm, to the 65th division on the other.

This kind of scale is constructed on the principle that lines drawn parallel to each other between the sides of an angle are proportional to the parts into which they divide the sides. (See Plane Geom., Theorems 61, 63, 65.)

The diagonal scale of equal parts is constructed as seen in the diagram, by parallel lines drawn along a ruler in sufficient number to embrace 10 spaces between them. Transverse lines are drawn perpendicularly to



these at intervals usually but not necessarily equal to the whole breadth of the 10 spaces. The last one of these intervals is divided into ten spaces, and the first division at top is joined to the second at bottom, the second at top to the third at bottom, and so on by diagonal lines.

To measure any distance on this scale, as 456, place the dividers on the 6th horizontal line from the top with one foot upon the 4th of the larger divisions, from the 1st on the right, and extend the other foot of the dividers, till it reaches to the 5th smaller division in the right hand square.

5. Before describing the protractor, which is an instrument for laying off angles, it will be necessary to explain the method of estimating the magnitude of angles.

In Geometry, it is shown that angles are proportional to the arcs included between their sides, the arcs being described with equal radii, and it is also there stated that hence such arcs are properly the measures of angles.

So that if an arc included between two sides of one angle be double, or triple, or sextuple, an arc described with the same radius included between the sides of another angle, the first angle is double, triple, or sextuple the second.

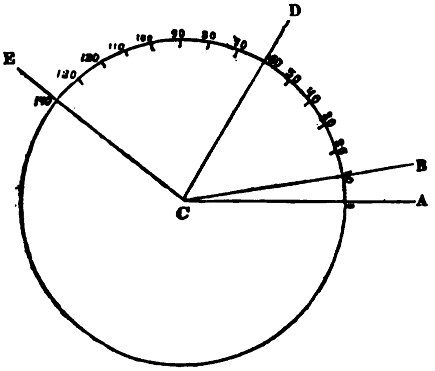
The relative magnitudes of angles may therefore be correctly expressed by means of the relative magnitudes of the arcs which measure them.

The relative magnitudes of quantities are commonly given by referring the quantities to be compared to some known standard of measure, which must be always of the same kind with the quantities themselves.

This standard is called a unit. Thus a foot, a yard, &c., are units of length, and the idea of the relative lengths of two lines is obtained by its being said that one is seven feet or yards, and the other nine. Or the just conception of the length of a *single* line is had by being told how many feet, yards, or miles it contains. The mind compares it with one of these well known units, which in imagination it repeats along its length.

Now the unit of measure, which is employed in a similar manner for giving the conception of the magnitude of an arc, is called a degree. A degree is the $\frac{1}{360}$ part of the circumference of a circle. The relation which any given arc bears to the whole circumference may be conveniently expressed by stating the number of degrees which the arc contains. Thus an arc of 90 degrees will be one fourth the whole circumference. An arc of 45 degrees will be one eighth. An arc of 30 degrees will be somewhat less. And it is plain that the length of the arc, as compared with the whole circumference, may be readily conceived, as soon as the number of degrees which it contains is mentioned.

So also the magnitude of the angles subtended by these arcs will, after a little familiarity, be rendered easily sensible to the mind. To speak of an angle of 10 degrees for instance (A C B in the annexed diagram), will suggest the image of a very acute angle, one of 60 degrees (A C D) a much larger acute angle, one of 140 degrees (A C E) an obtuse angle.



A degree being always the $\frac{1}{360}$ part of a circumference, a single degree will be larger in a larger circle than in a smaller, and this, so far from being inconvenient, is particularly advantageous in the measurement of angles; for since arcs described about the vertex of an angle as a centre with different radii, and included between the sides of the angle, bear the same relation to each other as the radii, and since the entire circumferences are also proportional to their radii, it follows that two concentric* arcs included between the sides of the same angle, and having the vertex of that angle for a centre, are the same aliquot parts of their respective circumferences. Consequently, two such arcs will contain the same number of degrees. Hence, to find the number of degrees contained in a given angle, the arc described for the purpose about the vertex, and extending from side to side of the angle, may be with any radius at pleasure.

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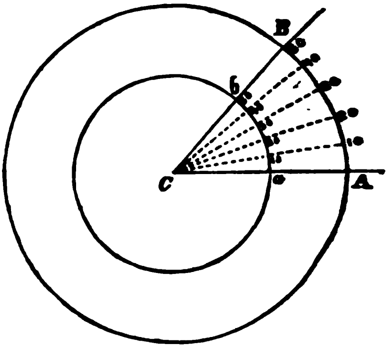
* Having the same centre.

This may be distinctly seen in the following diagram.



ACB is the angle; the larger arc AB included between its sides contains 50 degrees of the whole circumference; the arc *ab* with the lesser radius also contains 50 degrees, and so would an arc included between the sides of the given angle described with any other radius whatever.

Where the size of an angle is such that it does not embrace an exact even number of degrees of the circumference, smaller divisions called minutes, 60 of which make a degree, are employed. The angle is then said to contain as many degrees and minutes as there are degrees and parts of a degree, each $\frac{1}{60}$ over, between its sides. If the second side of the angle does not pass exactly through one of these smaller divisions, a still smaller kind termed seconds, 60 of which form a minute, or 360 a degree, must be introduced. More minute divisions than these last are not ordinarily used. When it becomes necessary to regard such, the same system is continued. The next denomination is thirds, 60 of which make a second; the next fourths, and so on.*

The notation for these denominations is as follows. Degrees are written thus $^\circ$; minutes thus $'$; seconds thus $''$; thirds thus $'''$, &c; $30^\circ\ 20'\ 10''$ is read thirty degrees, twenty minutes, and ten seconds.

6. It is evident that the numbers used in the system of division, for the circumference of the circle, are entirely arbitrary. Others might be employed with equal propriety, provided the same principles were observed. In fact the attempt has been made, and probably will be successful in France, to subvert the old system of division, and to adopt a decimal system in this as well as in every other sort of measurement. Thus a right angle, which is the unit of angles, is made to contain 100° instead of 90; and the circumference will then contain 400° instead of 360. $100'$ instead of $60 = 1^\circ$ $100'' = 1'$. Degrees in the centesimal division of the circumference are called *grades;* and the notation in this division is $^{g}\ ^{\backprime}\ ^{\backprime\backprime}$. Degrees are converted into grades by multiplying by $\frac{90}{100}$ or ·9. The convenience of a decimal division we have experienced in

---

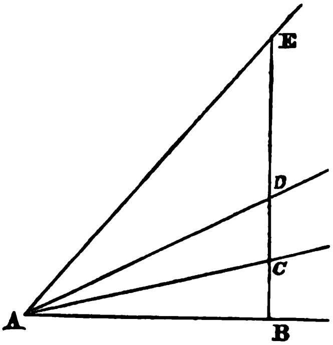
* Instead of thirds, fourths, &c., the almost universal practice now is to use decimals of a second, viz. tenths, hundredths, and thousandths.

this country in our system of Federal money. The two methods above described are called the sexagesimal and the centesimal divisions.

EXERCISES.

1. Convert $42^{g}\ 34^{\backprime}\ 56^{\backprime\backprime}$ or $42^{g}{\cdot}3456$ into degrees, &c.
   Ans. $38^{\circ}{\cdot}11104$ or $38^{\circ}\ 6'\ 39''{\cdot}74$.
2. Convert $24^{\circ}\ 51'\ 45''$ into grades, &c.
   Ans. $27^{g}\ 62^{\backprime}\ 50^{\backprime\backprime}$.
3. Prove $45^{\circ}\ 15'\ 20''=50^{g}\ 28^{\backprime}\ 39^{\backprime\backprime}{\cdot}50$.
4. Also, $10^{\circ}\ 15'\ 46''=11^{g}\ 40^{\backprime}\ 3^{\backprime\backprime}{\cdot}09$.
5. Also, $18^{\circ}\ 10'\ 48''=20^{g}\ 20^{\backprime}$.
6. Also, $\dfrac{180^{\circ}}{\sqrt{3}}=115^{g}\ 47^{\backprime}$.

7. Another method of expressing the magnitudes of angles is as follows. A distance at pleasure is laid off from the vertex of the angle upon one of the sides, and a perpendicular there drawn to this side till it meets the other side of the angle. The ratio of this perpendicular to the distance from its foot to the vertex, serves to indicate the size of the angle.



For example, if the line BCDE be perpendicular to the line AB, and BC be one fourth AB, the angle BAC is said to be an angle of $\frac{1}{4}$. If BD be one half AB, the angle BAD is said to be an angle of $\frac{1}{2}$. If BE be equal to AB, BAE is said to be an angle of 1; and so on for other magnitudes. An angle of 1 is plainly half a right angle, or 45°.

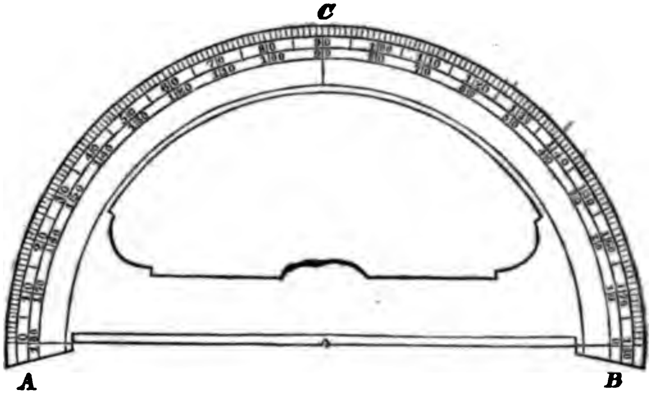
This kind of measurement is much used by engineers, to express the degree of slope in excavations and embankments.

8. The protractor which we are now prepared to describe is an instrument for drawing upon paper an angle of any given number of degrees.

This instrument is made in a variety of forms; sometimes with a full circle divided into degrees, sometimes comprising only a semicircle, sometimes upon a rectangular rule having not the circumference but the radii drawn, as they would be through the divisions of the circumference if it were actually described. The first kind is made usually of brass.

It has a metallic radius movable about the centre of the circle, and extending beyond the circumference. This prolonged radius serves to point out the number of degrees, and is armed with a sharp pin under the outer extremity for the purpose of pricking the paper, so that when the instrument is removed a line may be drawn with pencil through this point, and that upon which the centre was placed.

THE SEMI-CIRCULAR PROTRACTOR,



which is the one most commonly seen, is a semicircle of brass (or other metal), having the greater part of the interior cut out to render the instrument less heavy.

The semi-circumference is divided into degrees by marks made in the metal, and these are numbered from 0° to 180 (the number in a semi-circumference) both ways, in order that the counting may commence with convenience at either end.

The degrees are also sometimes divided into half degrees, and lines of different length are employed to mark more distinctly every five and every ten degrees.*

The centre is marked by a notch in the straight side of the instrument, which side is a diameter of the semicircle.†

9. In order to explain the use of the instrument here described, suppose it be required to draw at the point A in the line AB a line making with AB an angle of 22°.

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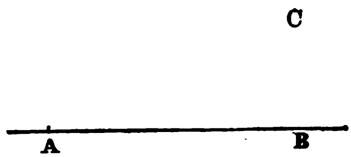
* Such a division of instruments is termed graduation.

† This instrument may be made out of paper, and a large one so made is very accurate.

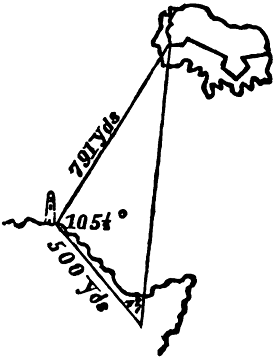


Place the protractor so that its centre shall be upon the point A, and its straight edge or diameter upon the line AB. Then mark the paper at the point C against the 22d division of the protractor, and a line joining C and A will form with AB the angle required.

10. We are now prepared to construct triangles when three of their six elements are given, the angles in degrees and the sides in feet, yards, or other linear units.

In order to show the practical utility of trigonometry at the same time that we explain the solution of a triangle, let us take the following problem in the calculation of distances to inaccessible objects.

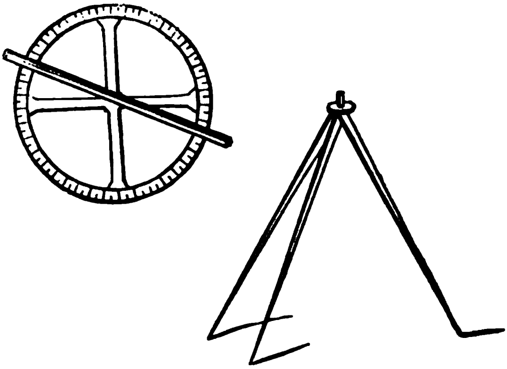
Suppose a fort situated upon an island, and a light-house upon the main shore, and let the distance from the light-house to the nearest salient of the fort be required.



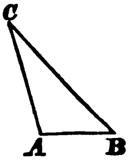
Measure a line along the shore of any length at pleasure, say 500 yards, beginning at the light-house. Then if two lines be imagined to be drawn from the extremities of the line just measured, to the salient of the fort, a large triangle will be formed having its two longest sides resting upon the sea. If now the angles which these two sides form with the first side, which we will call the base, could be determined by observation upon the shore, there would be known in this triangle a side and the two adjacent angles, which would be sufficient data to construct the triangle on a small scale, and to obtain the length of the required side extending from the light-house to the salient of the fort.

A somewhat rude instrument for the purpose of observing such angles as those alluded to above, might easily be made.

Let there be a circle, or flat circular ring of wood, divided into degrees, and having a tin tube movable upon a pivot at the centre of the circle; the tube being closed at one end except a very small orifice, and having two threads crossing at right angles in the centre of the other end, so that in looking through the tube with the eye at the small orifice, the line of sight may coincide with the axis. Let this apparatus be mounted upon a three-legged stand called a tripod, so that the plane of the circle shall be horizontal; then,



by placing the instrument thus formed at the light-house, in the example above, and sighting with the tube, first to a staff at the other extremity of the base, and then to the salient of the fort, keeping the circle stationary, the number of degrees passed over upon its circumference by the tube will indicate the angle of the triangle at the lig[illegible]house. This angle we shall suppose to be 105½°. The angle at the other extremity of the base might be found in the same manner, and suppose it 47°.[*]



To construct the triangle with these data, draw on paper a line AB, and make it equal in length to five hundred divisions of some scale of equal parts.[†] Then draw an indefinite line AC making with AB an angle of 105½°. Also lay off in a similar manner at the point B an angle of 47°; the two lines AC and BC will meet at C. Take the line AC in the dividers and apply them to the scale. The number of equal parts upon the scale between the feet of the dividers, will show the number of yards from the light-house to the fort. The number is 791.

If the angle at C were required, it might be measured by applying to it the protractor; or it is equal to $180° - (A + B) = 27\frac{1}{2}°$.

The side B C, if among the sought parts, might also be measured from the scale.

11. The instrument described above may be rendered suitable for application to the determination of heights. If a round bar be made to project horizontally from the top of the tripod, so that the graduated circular frame can be suspended by the socket at its centre in a vertical position, it will then serve to measure angles in a vertical plane.[‡]

---

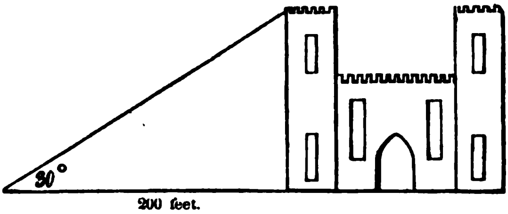
[*] The instrument here described is of course very rude. It was deemed advisable to postpone a description of more accurate instruments to a subsequent part of the work.

[†] This may be done conveniently by taking 50 divisions, and considering each division as equal to ten.

[‡] A vertical plane is one perpendicular to the surface of the earth.

To show the use of the instrument thus prepared take the following problem.

Required the height of a tower which stands upon horizontal ground, and the base of which is accessible.



Measure back a distance from the base of the tower, say 200 feet; call this distance the base line; at the extremity of the base line place the instrument arranged for taking vertical angles: suspend a plumb line from the centre of the circle, and the point 90° distant from that in which the plumb line cuts the circumference will be the point through which a horizontal radius would pass. Then sight with the tube to the top of the tower: the number of degrees between the tube and the horizontal radius just mentioned, will be the measure of the angle included between a line drawn to the top of the tower and the base line; let this number be 30°. Constructing a right angled triangle upon paper, having its base 200 and angle at the base 30°, the perpendicular of this triangle, measured by a scale of equal parts, will be the height of the tower. The height of the instrument must be added to the result found.

N. B. The sides found will always be expressed in units of the same kind as the base.

12. It is evident that when *any* three parts of a triangle, one of which is a side, are given, the other three may be discovered by a process similar to those just exhibited.

This kind of solution is said to be by construction.

The accuracy of the results must depend upon the niceness of the instruments, and the care with which the construction is made.

A degree of accuracy so uncertain and so variable is quite inadequate for many purposes to which Trigonometry is applied.

A method of *calculating* the required from the given parts of a triangle, which should produce always the same results from the same data, and be either perfectly, or so nearly exact, as to leave an error of no importance, however great the dimensions employed, would be evidently a desideratum. Such a method we have, and it is that which it will be the object of the residue of the present treatise to unfold.

To give the student a general view of what is before him, it will be well to state that a number of equations will be found, each containing four quantities, which quantities will be general expressions for the measures of elements of a triangle. The equation will express the true relation between these elements. By making one of these elements the unknown quantity and resolving the equation with respect to it, its value will be expressed in terms of the other three. If now these three were given, the value of the fourth would be known the moment the values of the three given were substituted for their general representatives.

It is plain that as many such general equations will be required, as there can be formed essentially different combinations of four out of the six elements of a triangle.

Equations like those here alluded to are called formulas, because each is a general form, under which a multitude of particular examples are included.

As these general formulas require of necessity the use of algebraic symbols and processes, and as *algebra*, from its power and application to decompose combinations of quantity so as to extricate their elements, is often called *analysis*, the subject upon which we are now about to enter is called

ANALYTICAL TRIGONOMETRY.

13. The sides and angles of a triangle are not quantities of a similar kind, and therefore do not admit of direct comparison. Since angles are expressed in degrees, and sides in units of length, one of the first principles which governs the formation of equations, namely, that the members and terms should express quantities of the same kind, would be violated by the introduction of angles and sides together, without some modification of one or both.

The expedient which has been invented to accommodate these heterogeneous quantities to each other, is that of employing straight lines, so related to the arcs which measure the angles of a triangle, as to depend upon these arcs for their length, in such a manner that when the arcs are known, these straight lines may be known also; and vice versâ. The chords of arcs are plainly lines of this description, and chords were at one time used for the purpose of which we here speak; but there is a more convenient kind of lines, of which there are three principal sorts, termed sines, tangents, and secants, of an arc or angle, called, when spoken of collectively, *trigonometrical lines*, the nature and use of which we shall presently explain. These lines being straight and expressed, as they will be found to be, in linear dimensions, like the sides of a triangle, they may be employed with the latter in equations or formulæ; and when, by the resolution of an equation of this description, one of these trigonometrical lines is found in terms of one or more sides of the triangle, the angle to which the trigonometrical line belongs may also be supposed to be known. How the former is known from the latter will be hereafter explained. Let it be taken for granted here that the knowledge of a trigonometrical line is equivalent to the knowledge of its arc or angle, and vice versâ.

The trigonometrical lines are sometimes called trigonometrical functions* of an arc or angle.

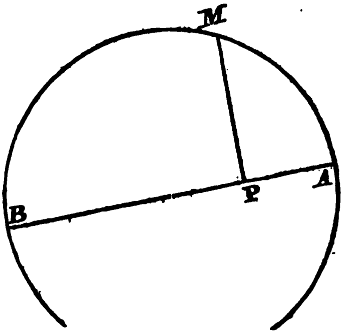
Of these trigonometrical lines, we now proceed to explain the nature and properties.

THE SINE.

14. The sine of an arc is a perpendicular let fall from one extremity of the arc upon the diameter drawn through the other extremity.

Thus the line MP is the sine of the arc AM.

The same line MP is likewise the sine of the arc BM, because it is a perpendicular let fall from one extremity M of the arc upon the diameter drawn through the other extremity B.



15. Two arcs, which together make a semi-circumference, have, it thus appears, the same sine. Two such arcs are called supplements of each other. A semicircle contains 180°. The supplement of an arc is therefore what is left after taking the arc from 180° or 200^{g}. Thus 80° is the supplement of 100°. 70° is the supplement of 110°. 85^{g} is the supplement of 115^{g}; in general $90^{c}—a$, or $200^{g}—a$ is the supplement of the arc a.

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* One quantity is said to be a function of another, when the former depends in any way upon the latter for its value. It is said to be an *increasing* function when it increases as the quantity upon which it depends increases; and a *decreasing* function when it diminishes as the other increases. The latter is called the argument.

EXERCISES.

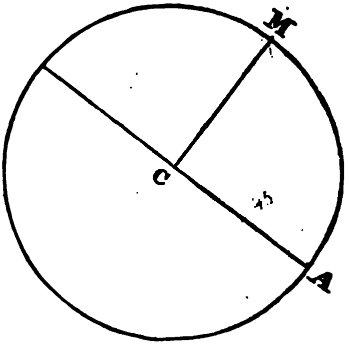
1. The supplement of $56^\circ\ 20' = 123^\circ\ 40'$.
2. " $186^\circ\ 12' = -(6^\circ\ 12')$.
3. " $37^\circ\ 4'\ 3'' = 142^\circ\ 55'\ 57''$.
4. " $115^\circ\ 13'\ 24''{\cdot}66 = 64^\circ\ 46'\ 35''{\cdot}34$.
5. " $226^\circ\ 14'\ 17'' = -(46^\circ\ 14'\ 17'')$.
6. " $23^{g}\ 25^{\backprime} = 176^{g}\ 75^{\backprime}$.
7. " $110^{g}\ 40^{\backprime}\ 50^{\backprime\backprime} = 89^{g}\ 59^{\backprime}\ 50^{\backprime\backprime}$.

Two arcs, then, which are supplements of each other, have the same sine, or, as it is sometimes expressed, the sine of an arc is equal to the sine of its supplement.

If a represent an arc of any number of degrees, the notation employed to express the sine of that arc is sin a. The proposition* above, stated algebraically, will stand thus, $\sin\ a = \sin\ (180^\circ - a.)$

The sine of an arc is also the sine of the angle measured by that arc.

16. When the arc is very small, it is plain that its sine will be very small also, and that when the arc is 0, the sine will be 0.



As the arc increases the sine increases till the arc is 90°, which, being a quarter of the circumference, is called a quadrant, the sine of which is R. (R signifying radius; which line this letter, whenever employed hereafter, will be understood to represent.)

As the arc increases beyond 90°, the sine diminishes, i. e. becomes a decreasing function of the arc, till the arc reaches 180°, when the sine is 0 again. Beyond this value of the arc the sine again increases till the arc reaches 270°, or three quadrants, when the sine is again equal in length to R.

From 270° to 360° the sine decreases, till at the latter value it is a third time 0. Beyond 360° we pursue the same round again, and no new variations are developed.

17. The least value of the sine is 0. It has this value at 0°, at 180°, and at 360°.

The greatest value of the sine is R. It has this value at 90° and at 270°. It has all possible values between 0 and R, but it has no different values, as the arc increases to two, three, and four quadrants, from those which it had in the first. So that when the sine of an arc greater than 90° is required, an arc, having an equal sine, may be found in the first quadrant. To find this arc we have the following rule, the correctness of which the annexed diagram will show. Observe how many degrees distant the termination of the given arc is from 180° or 360°, according to which of these two is nearest, and that number of degrees and fractions of a degree, will be the arc in the first quadrant, having the same sine as the given arc.

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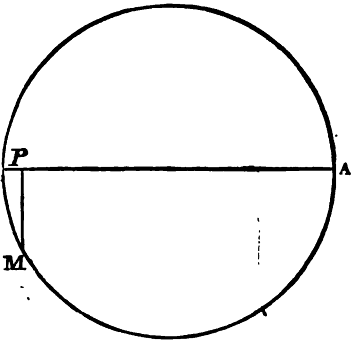
* The word proposition is here used in the enlarged sense of anything propounded as true.

For example, let the given arc be 200°. This is nearest 180°, and differs 20°. The sine of 20° is equal in length to the sine of 200°. Or M P, which is the sine of A B M, is also the sine of B M.



Again, let the given arc be 300°. This is nearest 360°, and differs 60°. The sine of 60° is equal in length to the sine of 300°.

If the given arc exceeds 360°, subtract 360, and then apply the rule just given. If the arc contains a number of circumferences, divide by 360, and apply the rule to the remainder.

18. It is customary, for the purpose of being able to bring the trigonometrical lines as they appear in the figure, the more readily before the mind when the figure is not present, to begin all arcs at the same point; and the point commonly chosen is the extreme right of the circumference, determined by the intersection of the horizontal diameter of the circle with the circumference. This is the point A, in the last figure. An arc of 90° will then reach to the top of the circle, or the upper extremity of a vertical diameter. An arc of 180° will terminate at the left of the circle, or of the horizontal diameter. An arc of 270°, at the lowest point of the circle, or lower extremity of the vertical diameter. An arc of 360°, at the right of the circle, or point of beginning.

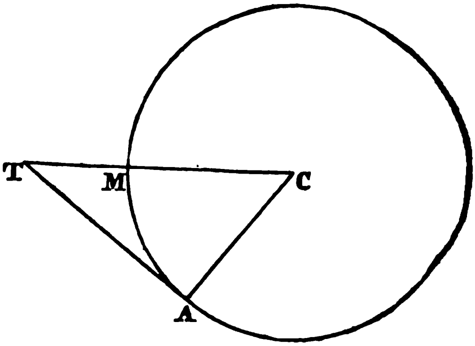
One advantage of this plan will readily appear. Since the arc always commences at the same point, namely, the right of the circle, the horizontal diameter will be the diameter which passes through one extremity of the arc, and wherever the arc may terminate, the perpendicular from the other extremity of it, which is the definition of the sine, will be a perpendicular to the horizontal diameter; so that the sines of all arcs, in a diagram so constructed, will be perpendiculars to the horizontal diameter.

The sines of arcs between 0° and 180° will be drawn downwards; and those of arcs between 180° and 360° will be drawn upwards.

According to the general principle of analysis, that quantities estimated in a contrary sense are distinguished by contrary signs, if the sines of arcs between 0° and 180° be considered as positive, those of arcs between 180° and 360° must be regarded as negative.*

THE TANGENT.

19. The tangent of an arc is a perpendicular drawn to the radius at one extremity of the arc, and terminated by the radius produced, which passes through the other extremity.



In the annexed diagram A T is the tangent of the arc A M. It is also the tangent of the angle A C M, measured by the arc.

The shorter the arc is, the shorter will be the tangent. When the arc is 0, the tangent will evidently be 0. As the arc increases, the tangent increases, and very rapidly as the arc approaches 90°. In order to trace the tangent through its various changes, we shall suppose the arc to commence at the point on the extreme right of the circle, and the degrees to be counted upwards, towards the left, as in a former case—the tangent of every arc will then be drawn at the extremity of the horizontal radius on the right of the centre, and be terminated by the radius produced, passing through the other extremity of the arc, which extremity will vary its position as the arc varies its magnitude.

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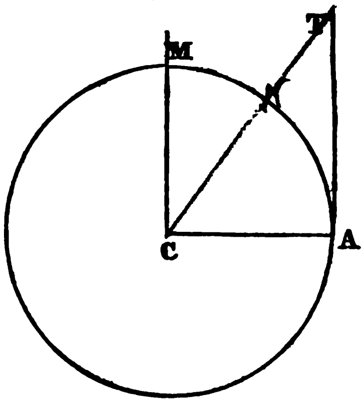
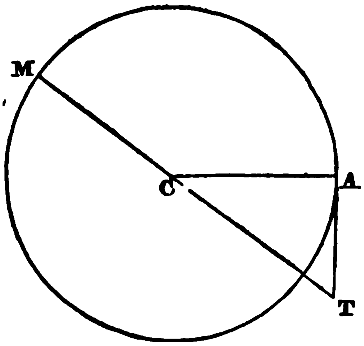
* See Algebra, page 182.

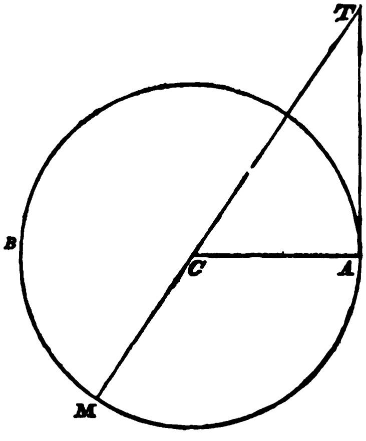
When the arc is 90°, the perpendicular to the radius at one extremity is parallel to the radius through the other extremity. These lines will never meet, and the tangent will have no termination. It is in this case said to be infinite. The sign employed to express infinity ∞ is also called the sign of impossibility. The value of the tangent of 90° is expressed algebraically thus, tan. 90 = ∞.



The tangent of an arc, terminating in the second quadrant, will be cut off below the origin* of the arc. Thus A T is the tangent of A M; and according to the principle adopted when treating of the sine, this tangent, being in the opposite direction to that of the tangent of an arc in the first quadrant, is negative.



When the arc is 180°, the negative tangent, which became shorter and shorter as the second extremity of the arc approached this point, again reduces to 0. Beyond 180°, or in the third quadrant, the tangent is cut off above the origin again. Thus A T in the annexed diagram, is the tangent of the arc A B M. The tangent of an arc in the third quadrant is, therefore, positive. When the arc is 270° or 3 quadrants, the tangent becomes parallel to the radius which produced



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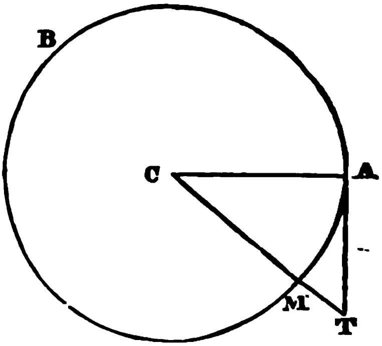
* A term applied to the point A, where the arc commences.

ought to terminate it, and the tangent is again ∞.*

The tangent of an arc in the fourth quadrant is negative, as may be seen from the annexed diagram.



20. The least value of the tangent is 0. The greatest value is ∞. So that the tangent has all possible values. But these it has, if we do not regard the sign, in the first quadrant; and the same rule applies to finding the length of the tangent belonging to any given arc, from that of an arc in the first quadrant, as was given for the sine.

The tangent changes its sign in every quadrant, that is four times in going round the circle. It is positive in the first and third, two diagonal quadrants, and negative in the second and fourth, the other two diagonal quadrants.

The tangent is ∞ at the top and bottom of the circle, and 0 on the right and left.

THE SECANT.

21. The secant of an arc is a line drawn from the centre of the circle to the extremity of the tangent.

In the preceding diagrams, CT is the secant of the arc AM. It is also the secant of the angle measured by the arc.

As the arc with its tangent diminishes, the secant diminishes; and when the arc and tangent are 0, the secant is equal to R. The secant can never be less than radius, because the tangent cannot pass within the circumference, and consequently the line from the centre to the extremity of the tangent, must extend at least to the circumference. When the arc is 90° the secant is ∞. When the arc is 180° the secant is R. And when the arc is 270° or three quadrants, the secant is again ∞. All which will appear from an inspection of the last diagrams.

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[*] The infinity here has the doubtful or double sign $\pm\infty$. Zero may have always the double sign ± 0. Infinity only when it is the transition from a + to a — value or vice versâ. (See Art. 36.)

The tangent and secant have their greatest values, namely ∞, together; that is, at the top and bottom of the circle. They have also their least values, that of the tangent being 0, and that of the secant R, together, to wit, at the right and left points of the circle.

22. In the first quadrant the secant is estimated from the centre towards the second extremity of the arc. In the second and third quadrants it is estimated in the opposite direction. According to the principle which it is necessary to observe, and of which we have before spoken, the secant must in these quadrants be considered as negative. In the fourth quadrant the secant is again estimated towards the second extremity of the arc, and is therefore positive.

The vertical diameter separates the positive from the negative secants, the positive being in the quadrants on the right of this diameter, and the negative being on the left.

23. We have now exhibited three of the trigonometrical lines. There are three others closely connected with these in character, called the cosine, the cotangent, and the cosecant; the reason for which names will presently appear.

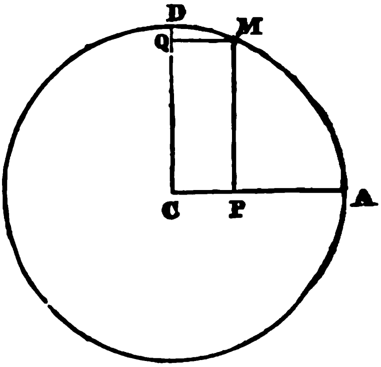
The difference between an arc or angle and a right angle or $90^\circ = 100^g$, is called the complement of the arc or angle. Thus 40° is the complement of 50°; 60° is the complement of 30°; 75^g is the complement of 25^g, and in general $90^\circ - a$, or $100^g - a$, is the complement of the arc a.

EXERCISES.

1. The complement of $24^\circ\ 32' = 65^\circ\ 28'$.
2. " $110^\circ\ 15' = -(20^\circ\ 15')$.
3. " $17^\circ\ 36'\ 43'' = 72^\circ\ 23'\ 17''$.
4. " $29^\circ\ 27'\ 6''\cdot 32 = 60^\circ\ 32'\ 53''\cdot 68$.
5. " $216^\circ\ 45' = -126^\circ\ 45'$.
6. " $65_g\ 34^{\backprime}\ 27^{\backprime\backprime} = 34^g\ 65^{\backprime}\ 73^{\backprime\backprime}$.
7. " $107^g\ 44^{\backprime}\ 20^{\backprime\backprime} = -7^g\ 44^{\backprime}\ 20^{\backprime\backprime}$.

The cosine, cotangent, and cosecant, are the sine, tangent, and secant of the complement. Thus the cosine of 50° is the sine of 40°; the cotangent of 30° is the tangent of 60°; and in general the cosine, cotangent, or cosecant, of the arc a, is the sine, tangent, or secant of $90^\circ - a$.

THE COSINE.



24. In the annexed diagram DM is the complement of the arc AM; and MQ being a perpendicular from one extremity M of the arc DM upon the diameter which passes through the other extremity D, is the sine of the arc DM. Therefore by the definition it is the cosine of the arc AM. But MQ = CP. Hence CP is also the cosine of the arc AM. We have then another definition for the cosine of an arc, viz. *the distance from the foot of the sine of the arc to the centre of the circle.*

25. If the arc terminate on the right of the vertical diameter, i. e. in the first or fourth quadrant, the foot of the sine will fall on the right of the centre; but if the arc terminate on the left of the vertical diameter, i. e. in the 2d or 3d quadrant, the foot of the sine will fall on the left of the centre. The cosine being estimated in opposite directions in these two cases, must have opposite signs. It is therefore positive in the 1st and 4th quadrants, and negative in the 2d and 3d.

It will be recollected that the positive were separated from the negative secants, as the positive are here seen to be from the negative cosines, by the vertical diameter. The secant and cosine have therefore always the same algebraic sign.

It was shown (Art. 15), that sin $(180^\circ - a) = \sin a$; so also cos. $(180^\circ - a)$ is equal in length to cos. a, since they are both the distance from the foot of the same sine (MP in the diagram of Art. 14) to the centre, i. e. if we suppose one of the arcs to originate at A, the other at B, and both to be extended towards M in opposite directions.* But if $a < 90^\circ$, it follows that $180^\circ - a$ terminates in the second quadrant, hence its cosine is negative; if $a > 90^\circ$ then cos. a is negative, and $180^\circ - a$ being in the first quadrant, its cosine is positive; therefore, *the cosine of an arc and the cosine of its supplement are equal with contrary signs.*

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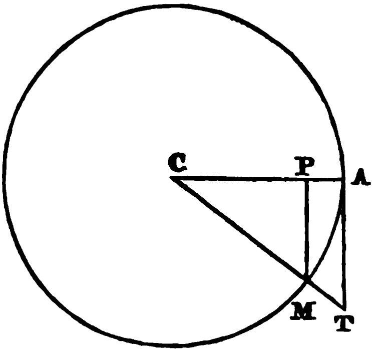
[*] Both arcs a and $180^\circ - a$ are now supposed to originate at the same point A, and to be estimated in the same direction.

26. The cosine of 0° (being equal to the sine of the complement of 0° which is 90°) is R. The cosine of 90° is equal to the sine of 0°, which is 0. The cosine of 180°, being the distance from the foot of the sine to the centre, and being also on the left of the vertical diameter, is — R, as may be seen from the preceding diagram. The cosine of 270°, being the distance from the foot of the sine to the centre, since the sine falls on the centre, is 0.

The least value of the cosine is 0; the greatest value is R. When the sine has its least value, the cosine has its greatest; and vice versâ.

The versed sine of an arc, which is seldom employed in Trigonometry, but often in Mechanics, is the distance from the foot of the sine to the origin of the arc, thus PA in the last diagram is the versed sine of the arc AM.

27. Before noticing the cotangent and cosecant, let us consider the manner of treating negative arcs. Such arcs commencing at the point A in the diagram ought evidently, on the general principle already repeatedly mentioned, to be laid off upon the circumference in the opposite direction from the positive arcs, i. e. downwards.



Let us for simplicity suppose the arc in question to be less than a quadrant; being laid off downwards, such an arc will terminate in the fourth quadrant. Hence we see that the trigonometrical lines of a negative arc must be affected with the same signs as those of an arc in the fourth quadrant. Thus *the sine of a negative arc will be —, the cosine +, the tangent —, the secant +.*

Secondly, suppose the given negative arc to be greater than a quadrant; were it positive, some of its trigonometrical lines would be negative. The rule given above, which determines the signs of its trigonometrical lines, by those of an arc in the 4th quadrant, will apply with this modification, that *when the trigonometrical line is + in the fourth quadrant, the corresponding trigonometrical line of the negative arc has the same sign as that of a positive arc of the same magnitude, and when the trigonometrical line is — in the fourth quadrant, a contrary sign.*

The truth of this assertion may be seen, by trying negative arcs of various magnitudes upon the diagram, laying them off downwards from the right point of the circle, and observing in which quadrant their extremities fall. They will be found in every case to give results agreeable to the rule just stated.

THE COTANGENT AND COSECANT.

28. The cotangent of 0° is equal to the tangent of 90° (Art. 23) and is therefore ∞. The cotangent of 90° is equal to the tangent of 0° and is 0. The cotangent of 180° is equal to the tangent of $90° - 180° =$ the tangent of $-90° = \infty$, since $-90°$ is a negative arc, and terminates at the bottom of the circle, or the 270° point. The cotangent of $270° =$ the tangent of $90° - 270° =$ the tangent of $-180° = 0$.

When the tangent has its least value, which is 0, the cotangent has its greatest which is ∞, and vice versâ.

29. The cosecant of $0° =$ the secant of $90° = \infty$. The cosecant of $90° =$ the secant of $0° = \text{R}$. The cosecant of $180° =$ the secant of $90° - 180° = \infty$. The cosecant of $270° =$ the secant of $-180° = -\text{R}$.

When the secant has its least value, which is R, the cosecant has its greatest, which is ∞, and vice versâ. The cotangent and cosecant have their greatest values together and their least values together, viz. that of the one 0, of the other R, at the top and bottom of the circle, and both ∞ at the right and left points.

30. With regard to the signs of the cotangent and cosecant in the different quadrants, they will be most conveniently discovered from the analytical expressions for these lines which we shall presently have. We add here, however, which so far as the cotangent and cosecant are concerned must be for a moment taken for granted, that the six trigonometrical lines may be arranged in three pairs, each pair having always the same algebraic sign.

We have seen that the secant and cosine go together in this way; so do also the cosecant and sine; and so do the tangent and cotangent. The positive sines and cosecants are separated from the negative by the horizontal diameter; the positive cosines and secants from the negative, by the vertical diameter; and the tangent and cotangent are together + and — alternately in the successive quadrants.

31. The following algebraic notation is employed for the six trigonometrical lines. Let a be the algebraic expression for the number of degrees in any arc, then the trigonometrical lines of the arc a will be expressed thus; sin a, tan a, sec a, cos a, cot a. cosec a.

Cot a tan $a = R^2$ is read, the cotangent of the arc a multiplied by the tangent of the same arc is equal to the square of the radius of the circle in which these trigonometrical lines are supposed to be drawn. Cot a and tan a are expressions for straight lines, and the equation above expresses that the rectangle formed by the tangent and cotangent of an arc is equivalent to the square formed upon the radius.

The two members of the above equation contain the same number of dimensions, and are therefore homogeneous. This ought to be the case in all trigonometrical equations; because a line cannot be equal to the rectangle of two lines or a surface, nor either of these to a solid.

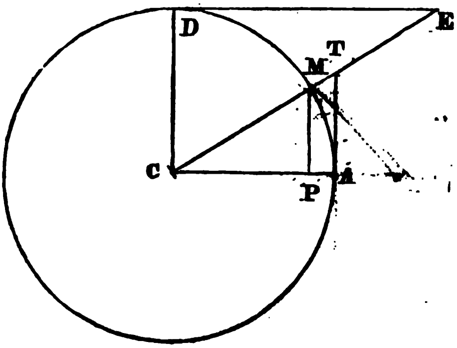
Sometimes in analytical investigations R is supposed to be equal to 1; R^2 and R^3 would also be equal to 1. Whether this 1 is a unit of length, of surface, or of solidity, must be determined by what is required to preserve the homogeneity of the equation.

32. The tangent, secant, cotangent, and cosecant may be expressed in terms of the sine and cosine.

The values of the four former in terms of the two latter are derived geometrically as follows:

Call the arc AM in the diagram, a; then $DM = 90° - a =$ complement of a, $DE =$ cot a and $CE =$ cosec a.

In the similar triangles CPM and CAT, since homologous sides are proportional, we have



$$CP : PM :: CA : AT$$

or

$$\cos a : \sin a :: R : \tan a$$

whence multiplying the means and dividing by the first term, we obtain the last

$$\tan a = \frac{R \times \sin a}{\cos a}$$

that is, *the tangent of any arc is equal to radius multiplied by the sine divided by the cosine of the same arc.* If R be made equal to 1, then

$$\tan = \frac{\sin}{\cos}$$

33. In the same similar triangles we have

$$\text{CP} : \text{CM} :: \text{CA} : \text{CT}$$

or,

$$\cos a : \text{R} :: \text{R} : \sec a$$

hence,

$$\sec a = \frac{\text{R}^2}{\cos\ a}$$

when R = 1

$$\sec = \frac{1}{\cos}$$

34. In the triangles CMP and CED, which have their sides respectively parallel, and are therefore similar, we have the proportion,*

$$\text{MP} : \text{CD} :: \text{CP} : \text{DE}$$

or,

$$\sin a : \text{R} :: \cos a : \cot a$$

whence,

$$\cot a = \frac{\text{R} \cos a}{\sin a}$$

when R = 1

$$\cot = \frac{\cos}{\sin}$$

35. The same triangles give also the proportion

$$\text{MP} : \text{DC} :: \text{CM} : \text{CE}$$

or,

$$\sin a : \text{R} :: \text{R} : \operatorname{cosec} a$$

whence,

$$\operatorname{cosec} a = \frac{\text{R}^2}{\sin\ a}$$

R being 1

$$\operatorname{cosec} = \frac{1}{\sin}$$

---

* The homologous sides are those which are parallel.

36. In the expressions for the tangent and cotangent which we have here derived, it will be observed that we have the quotient of the sine and cosine, and that therefore when the sine and cosine have contrary signs, the tangent and cotangent will be negative. This occurs in the second and fourth quadrants.

It appears hence, that the cotangent changes its sign always with the tangent.

Also that both the tangent and cotangent of an arc are equal to those of its supplement with contrary signs.

From the expressions for the secant and cosecant, it appears that the former must always have the same sign as the cosine, and the latter the same as the sine.

The formulas derived in the last four articles should be committed to memory.

Quantities in changing their signs pass through zero or infinity. (See Alg. Note 3d, p. 176.) Thus the sine changes from + to — or vice versâ, twice in going round the circle; viz. in passing through 0 at 0° and 180°; the cosine twice in passing through 0 at 90° and 270°; the tangent four times, in passing through 0 at 0° and 180°, and ∞ at 90° and 270°; the cotangent four times, in passing through 0 at 90° and 270°, and through infinity at 0° and 180°; the secant twice, in passing through ∞ at 90° and 270°; the cosecant twice, in passing through ∞ at 0° and 180°.

37. Multiplying the expression for the tangent given in Art. 32 by that of the cotangent in Art. 34, we have

$$\tan a \cot a = \text{R}^2$$

whence,

$$\tan = \frac{\text{R}^2}{\cot}$$

and,

$$\cot = \frac{\text{R}^2}{\tan}$$

when R = 1 the above expressions become

$$\tan = \frac{1}{\cot}$$

$$\cot = \frac{1}{\tan}$$

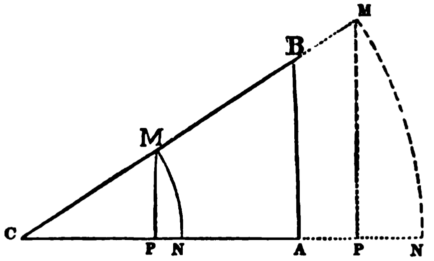
i. e. the tangent and cotangent are reciprocals of each other.

EXERCISES.

1. Express each of the six trigonometrical lines in terms of each one of the other five by itself.
2. The sine of an angle being 0·856, find the other trigonometrical functions.
3. The tangent being 2·34, find the others.
4. The cotangent being 1·203.
5. State the equivalents of the following functions of angles greater than 90°, or obtuse angles in equivalent functions of angles less than 90° or acute angles, sin 170°, cos 147°, tan 98° 31′, cot 171° 14′, sec 171°, cosec 155°, sec 215^g, cos $318^g\ 10^{\backprime}$, tan $271^g\ 81^{\backprime}$, cot $204^g\ 18^{\backprime}\ 94^{\backprime\backprime}$.
6. Find the versed sine of the angle, the cosine of which is ·93358.
7. Prove the vers. = 1 — cos to be always positive.
8. Prove the greatest value of the versed sine to be 2R.

38. We are now prepared to find formulas for the solution of right angled plane triangles in all cases, and plane triangles in general in a few particular ones. The remaining cases of triangles in general will require further preliminary matter.

DERIVATION OF FORMULAS FOR THE SOLUTION OF RIGHT ANGLED PLANE TRIANGLES.



Let ABC be any right angled triangle. With C as a centre describe, with any radius at pleasure, the arc MN terminating at the sides of the angle. This arc will be the measure of the angle C. Draw MP perpendicular to CN. MP will be the sine of the arc MN because it is drawn from one extremity M of the arc perpendicular to the diameter which passes through the other extremity N.

MP is also the sine of the angle C. CM is the radius of the circle to which the arc MN belongs.

The two triangles CMP and CBA are equiangular and similar, and give the proportion

$$CM : MP :: CB : BA \text{ or } R : \sin C :: CB : BA.$$

Had an arc been described with B as a centre in a similar manner we should have had R : sin B : : BC : CA, from which it appears that *the radius of any circle whatever* bears the same proportion to the sine in that circle of the arc which measures one of the acute angles of a right angled triangle, that the hypothenuse of the triangle does to the side opposite the acute angle.*

It is customary, for conciseness, to represent the sides opposite the angles of a triangle by small letters of the same name with the large letters which are placed at the angles; which large letters are also employed as the algebraic representatives of the angles. Thus in the triangle above, A being the right angle, the hypothenuse opposite is expressed by a; the side AC opposite B is represented by b, and so the other. The above proportions would, according to this method, be written thus

$$\left.\begin{array}{l} \mathrm{R} : \sin \mathrm{B} :: a : b \\ \mathrm{R} : \sin \mathrm{C} :: a : c \end{array}\right\} \quad (1)$$

Both these proportions are expressed in the single rule printed in italics above. When $\mathrm{R} = 1$, multiplying the second and third terms, and dividing by the first, in the preceding proportions we have

$$b = a \sin \mathrm{B}$$

and

$$c = a \sin \mathrm{C} \quad (2)$$

That is *either perp. side = the hypoth.* × *the sine of the angle opposite.*

The two acute angles of a right angled triangle are together equal to a right angle or 90° (Plane Geom. Theorem 15, Cor. 5), therefore they are complements of each other; hence $\sin \mathrm{C} = \cos \mathrm{B}$; and the second of the above proportions (1) may be changed into

$$\mathrm{R} : \cos \mathrm{B} :: a : c \quad (3)$$

which may be translated into ordinary language thus; *radius* : *the cosine of one of the acute angles of a right angled triangle* : : *the hypothenuse* : *the side adjacent the acute angle.*

When any three terms of a proportion are given, the remaining term can be found. If the unknown term be one of the extremes, *multiply*

---

* It is important to observe that the same trigonometrical lines of angles or arcs containing the same number of degrees in two different circles bear the same relation to each other. Thus in the diagram above, CM : MP : CM' : MP, or,

(R : sin) of the smaller circ : : (R : sin) of the larger;

also,

CM : CP : : CM : CP or (R : cos) of the one : : (R : cos) of the other.

the two means and divide by the other extreme; if the required term be a mean, *multiply the two extremes, and divide by the other mean.*

When R = 1 we have from proportion (3) above

$$c = a \cos B \qquad (4)$$

i. e. *either perpendicular side of a right angled triangle, equal to the hypoth.* × *cos of the adjacent angle.*

The above formulas contain each of them two of the sides of a triangle, the sine or cosine of an angle, and radius. If the lengths of the sides be given in numbers, these numbers may be put in place of the small letters which represent the sides in the proportion, and the general form becomes so far adapted to a particular case in the solution of right-angled triangles; but if the angle be given in degrees, how are we to know its sine or cosine, for *that* is the quantity which enters into the formula; and how are we to know the numerical value of R? For the present the student must be satisfied with the reply, that he can find the numerical value of any trigonometrical line corresponding to an angle of any given number of degrees, in a table at the end of the work. This is Table XXIV.* of Natural Sines. The degrees for angles or arcs of every magnitude within the quadrant will be found at the top of the columns of the table, and the minutes in the column marked M on the left, if the given angle or arc be less than 45°; but if it be greater than 45°, the degrees will be found at the bottom of the page, and the minutes on the right; the length of the sine or cosine will be found in the column under or over the degrees, as the case may be, and on the same horizontal line with the minutes. The title of the column must be looked for at top if the arc be less than 45°, and at bottom if the arc be greater.† The other trigonometrical lines may be easily calculated from the sines and cosines, as will be seen in the examples.

The trigonometrical lines of this table are computed, by a rule which will be hereafter demonstrated, for a circle whose radius is 1. So far as the *principles* for the solution of triangles are concerned, the length of the radius is entirely immaterial, as it will be recollected that the arc in the last diagram was described with any radius at pleasure.

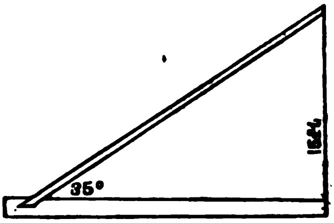
When, in cases of the solution of right angled triangles, the hypothenuse and one of the acute angles are either given or required by the problem, one of the above formulas is always employed.

---

* The tables are selected and printed from the stereotype plates of a very large collection.

† A decimal point must be understood at the left of all the numbers in the columns of the table entitled N. sine and N. cos.

39. Let us take an example by which to illustrate their application, and as upon a former occasion, one which shall at the same time exhibit the practical utility of Trigonometry.



A roof is to have a height of 15 feet in the interior at the centre, and an inclination of 35°.

Required the length of the inner line of the rafters.

A right angled triangle will be formed in which the angle at the base will be 35°, and the side opposite 15 feet, and of which the hypothenuse is required. Formula (1) of the last article applied to this case gives*

$$1 : \sin 35^\circ :: a : 15$$

Multiplying the extremes and dividing by the first mean the value of the other mean which is a, the hypothenuse required, will be obtained

$$a = \frac{1 \times 15}{\sin\ 35^\circ}$$

Or formula (2) by a simple transformation gives the same thing.

$$a = \frac{15}{\sin 35^\circ}$$

Looking out the sine of 35° in the tables and performing the operations indicated in the last equation, the value of a will be known, which will be the length of the rafters required. The answer will be in feet.

Sin 35° is found from the tables to be ·57358.

thus

$$a = \frac{15}{\cdot 57358} = 26.1 \text{ feet}$$

If (to vary the problem) half the interior breadth of the roof had been given, say 20 feet, and the angle of inclination, instead of 35° as in the last example, had been 15°, then to find the length of the rafters, it would be necessary to find first the angle opposite the given side 20 feet; which is done by subtracting the given angle 15° from 90°, since the two acute angles of a right angled triangle are complements of each other. The remainder is 75°.

---

* Since in the demonstration of the formulas, the sides and angles of the triangle were supposed to have no particular values, it follows that any numbers, or any other letters compatible with the properties of a triangle, may be put in the place of those employed, and the formulas will still be true. This must be borne in mind throughout the work.

Applying the same formula as before, there results the proportion

$$1 : \sin 75^\circ :: a : 20$$

or the equation

$$20 = a \sin 75^\circ$$

whence,

$$a = \frac{20}{\sin 75^\circ} = \frac{20}{\cdot 96600} = 20\cdot 7$$

The same result might be obtained by using the given angle 15°, and employing formula (3) above, which contains the cosine of one of the acute angles. The proportion would stand thus

$$1 : \cos 15^\circ :: a : 20$$

or its equivalent (4)

$$20 = a \cos 15$$

whence,

$$a = \frac{20}{\cos 15^\circ} = 20\cdot 7$$

the same as before. In fact $\cos 15^\circ = \sin 75^\circ$. (See Art. 23.)

40. Had the height and half the breadth of the interior of the roof been given, the length of the rafters might have been obtained, by employing the property of the right angled triangle demonstrated at Theorem 26, of Plane Geometry, that the square on the hypothenuse is equivalent to the sum of the squares upon the other two sides. Let the height of the roof be 12 feet, and the semi-breadth 16 feet, then

$$a^2 = 12^2 + 16^2 = 400$$

whence,

$$a = 20$$

If the length of the rafters had been given equal to 20 feet, and the height of the roof equal to 12 feet, then the semi-breadth would have been expressed thus

$$b^2 = 20^2 - 12^2 = (20+12)\,(20-12)^* = 32 \times 8 = 256$$

whence,

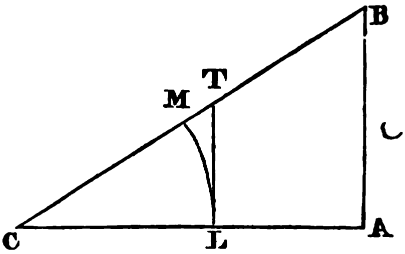
$$b = 16$$

---

* See Algebra, p. 16, ex. 2, and note.

41. Had the semi-breadth or base of the triangle and the inclination of the roof been given, and the height of the roof or perpendicular of the triangle been required, the hypothenuse not entering into the problem, neither of the above formulas, all of which contain the hypothenuse, would serve to find the side required in a direct manner. It might, however, be found indirectly by first finding the hypothenuse, using one of the above proportions, and then by means of the hypothenuse, using the same proportion, the required side might be obtained.

It is, however, objectionable to find one of the required parts in terms of the part which has itself been calculated from the given parts; because in the use of the tables which give the trigonometrical lines of the different angles not with perfect accuracy, but truly for as many decimal places as the table employs, a small error arises from the decimals neglected beyond the last place, and this, though so small as to be unimportant, becomes magnified by repetition, as in the case where one part of a triangle itself not perfectly accurate, is employed to calculate another. It is therefore desirable to find each of the required parts, in terms of the given parts; and this may always be done in right angled triangles. We proceed, therefore, to demonstrate a formula for the direct solution of the last case supposed above.



Let ABC be a right angled triangle. With any radius at pleasure describe an arc ML which shall be the measure of the angle C. At the point L draw a perpendicular LT to the line CL, terminating at the line CT. LT is evidently the tangent of the arc ML, since it is a perpendicular to the radius at one extremity of the arc, and is terminated by the radius which passes through the other extremity, according to definition of Art. 19. It is also the tangent of the angle C.

The equiangular and similar triangles CLT and CAB give the proportion

$$CL : LT :: CA : AB$$

or,

$$R : \tan C :: b : c \qquad (1)$$

If $R = 1$ this proportion gives

$$c = b \tan C \qquad (2)$$

or,

$$\tan C = \frac{c}{b} \qquad (3)$$

Let the angle of the roof in the above problem be 20° and the semi-breadth 25 feet, then

$$1 : \tan 20^\circ : : 25 : c$$

whence,

$$c = \frac{\tan 20^\circ \times 25}{1}$$

Had the angle B been used instead of C, the resulting proportion would have been

$$R : \tan B : : c : b \qquad (4)$$

Both proportions may be expressed together in common language thus: *Radius* : *the tangent of one of the acute angles of a right angled triangle* : : *the side adjacent that angle* : *the side opposite.*

This last rule applied to the problem at Art. 11, gives

$$1 : \tan 30^\circ : : 200 : c$$

whence,*

$$c = \tan 30^\circ \times 200 = {\cdot}57735 \times 200 = 115{\cdot}47000$$

c is the height of the tower.

The same rule will evidently serve to determine either of the acute angles of a triangle when the two perpendicular sides are given.

If the side c were given and the angle B, the side b might be found in the same manner, using the proportion (4) which contains the angle B.

42. We have now exhibited all the cases which can possibly occur in the solution of right angled triangles, with some specimens of their application. The right angle of the triangle is fixed; and any two of the five remaining parts being given, the other three may be found. Let the student select at pleasure any two of the five parts, the two selected to be considered as given, and he will find the case for solution with which he will then be presented, solvable by some one of the formulas above.

The operations in the cases already exhibited, though of the most simple kind, nevertheless involve multiplications and divisions, which, from the number of places of figures, are somewhat tedious. In more complicated cases this evil would be much increased.

---

* The tangent is found from Table XXIV. by dividing the sine by the cosine. (Art. 32.) Should the cotangent be required, divide the cosine by the sine. (Art. 34.) To find the secant divide 1 by the cosine. (Art. 33.) For the cosecant divide one by the sine. (Art. 35.)

On this account it is customary to employ in trigonometrical calculations, that ingenious invention of Lord Napier's for facilitating numerical calculations, the table of logarithms;[*] before explaining the use of which we shall give some exposition of the

THEORY OF LOGARITHMS.

43. The logarithm of any given number is the exponent of the power to which it is necessary to raise some particular number in order to produce the given number. Thus, let 10 be the number raised to the power; then 2 is the logarithm of 100, because $10^2 = 100$ and 3 is the logarithm of 1000, because $10^3 = 1000$. Every given number will have a corresponding logarithm or exponent of the power to which it is necessary to raise 10 in order to produce the given number.

The number 10, which is the only number that does not change in the above equalities, is called a constant.

Should the constant number which has been employed be changed for another, the logarithms of numbers would be different from those derived by the use of the first constant. Logarithms derived from different constants are said to belong to different systems of logarithms, and the constant number belonging to each system is called the base of that system. The system most in use has the number 10 for a base, and is called the common system. The relation which this number sustains to the decimal system of notation will readily suggest some reasons for its selection; it will be found, as we proceed, to have many advantages.

44. If b be the base of a system, n a number, and l its logarithm, then by the definition

$$b^l = n$$

If we put b in the place of n, this equation becomes

$$b^l = b$$

Here l is evidently equal to 1. Hence *the logarithm of the base of every system is* 1.

---

[*] Table XXVI. at the end. The tables most in repute are the French tables of Callet.

N. B. The tables in this volume having been printed from the accurate stereotype plates of Bowditch's tables, by permission of the proprietor, are numbered as in Bowditch's edition, and as but a part of his tables are necessary to the present work, the Nos. of the table must not be expected to occur in regular order.

If in the equation

$$b^l = n$$

n be made equal to 1, we have

$$b^l = 1$$

Here l is evidently equal to zero. (Algebra, Art. 17.) Hence *in every system*, *log. of* $1 = 0$.

45. Suppose now the system be the common system; b will be equal to 10. If we substitute for n all possible numbers successively, we shall have a series of equations like the following,

$$10^l = 1$$
$$10^l = 2$$
$$10^l = 3$$
&c.

In the first l is the common logarithm of 1, in the second of 2, in the third of 3, &c. If l be made the unknown quantity, and these equations be successively resolved, we shall have the common logarithms of all numbers.* If now a table be formed having the series of natural numbers, 1, 2, 3, 4, &c., in one column, and their logarithms calculated as above placed in a second column against them, this would be a table of logarithms. The tables in actual use do not differ from such an one in principle, though some arrangements are adopted in them to avoid unnecessary repetitions.†

In the common system the logarithm of 10 is 1, the logarithm of 100 is 2; and the logarithms of all numbers between 10 and 100 are between 1 and 2, that is, they are 1 and a fraction. The logarithm of 1000 is 3, and the logarithms of all numbers between 100 and 1000 are between 2 and 3, that is, they are 2 and a fraction. In the same manner it may be shown that the logarithms of all numbers between 1000 and 10,000 are 3 and a fraction; of all numbers between 10,000 and 100,000, 4 and a fraction, and so on. The logarithms of most numbers, therefore, are mixed numbers. The fractional part is written in the tables; the whole number part, which is called the characteristic, is not written, nor is it necessary that it should be; for numbers between 10 and 100, or those composed of two figures, have 1 for a characteristic, as has just been seen; numbers between 100 and 1000, or those containing three figures, have 2 for a characteristic; numbers containing four figures have 3 for a characteristic, and so on. Whence it appears that the characteristic is always 1 less than the number of digits in the number to which the logarithm belongs. So that if against any given number, the decimal part of its logarithm be found in the tables, the entire part or characteristic may be supplied by counting the figures in the given number, and making the characteristic, one less.

---

* The method of resolving them is given in Art. 325, Alg.

† For a more full exposition of the *theory* of logarithms, see Algebra, Art. 210, page 258, et seq.

In proceeding to explain the tables, we will premise that the logarithms of several consecutive numbers, if the numbers be somewhat large, will differ so little as to have several of their first figures the same. Hence, by a proper arrangement of the tables, the first figures of the logarithm may be written but once for several numbers, provided all be designated to which they refer, and thus much repetition be avoided.

The manner in which this is accomplished will be shown in the

EXPLANATION OF THE TABLES.

PROBLEM I.

46. *To find from the tables the logarithm of any given number.*

Case I.—*When the number is between* 100 *and* 10,000, if it be composed of three figures, find it in Table XXVI. at the end of the volume, and in the column at the left entitled No.; in the next column marked 0 at top and on the same horizontal line you will find the decimal part of the logarithm required. This contains five places.* If the given number contain four† figures, find the first three of it in the column No. as before, and the fourth in one of the columns marked 0, 1, 2, 3, &c., at top; under the latter, and on the same horizontal line with the first three, you will find the decimal part‡ of the logarithm sought.

N. B. The characteristic is always one less than the number of figures in the given number.

---

* In the tables of Callet seven places, the first three of which being the same for several numbers are not repeated, but must be understood before those which follow the number that has them expressed until you come again to seven places.

† In the tables of Callet substitute the word five for four in the above rule, and the word four for three.

‡ In the tables of Callet you find only the last four places, the first three to be prefixed to them must be taken from the numbers projecting to the left in the column marked 0 at top.

EXAMPLES.

1. Required the logarithm of 217.

In the column entitled No. on page 171 of Table XXVI. I find 217; in the next column marked 0 at top, and on the same horizontal line, I find 33646 for the decimal part of the logarithm required. The characteristic is 2, since 217 contains three figures, and the whole logarithm 2·33646.

2. Required the logarithm of 1122

On page 170 and in the column No. I find 112; in the column having the last figure 2 of the given number at top, and on the same horizontal line with the 112 before found, I find 04999 or with the proper characteristic 3·04999.*

3. Required the logarithm of 2188.

Ans. 3·34005.

47. We proceed now to show the use of logarithms in numerical calculations.

MULTIPLICATION.

Let b be the base of the system of logarithms, n any number, and l its logarithm. Then by the definition

$$b^l = n$$

Let n' be another number, and l' its logarithm, we have also

$$b^{l'} = n'$$

Multiplying these two equations, member by member, and observing the rule for exponents in multiplication, which is to add them together, we have

$$b^{l+l'} = nn'$$

From this last expression, it appears that $l+l'$ is the exponent of the power to which it is necessary to raise the base a, in order to produce the number nn'. But nn' is the product of n and n'. Hence the logarithm of the product is equal to the sum of the logarithms of the multiplier and multiplicand.

---

* At the tops of the pages in the table will be found catch numbers for the eye, in turning over the leaves, to show what numbers and logarithms are contained on the page.

EXAMPLE.

Multiply 2421 by 1613.
The logarithm of 2421 is 3·38399
The logarithm of 1613 is 3·20763

The logarithm of 2421 × 1613 or 3905073 is* 6·59162, or the sum of the logarithms.

48. If in addition to the numbers n and n' above, we suppose a third number n'' of which the logarithm is l'' we shall have in a similar manner

$$a^{l+l'+l''} = nn'n''$$

and so on.

Or, in general, *the logarithm of a product of several factors is equal to the sum of the logarithms of those factors separately.*

DIVISION.

49. Dividing the equation.

$$b^{l} = n$$

by the equation

$$b^{l'} = n'$$

we have, observing the rule of division, to subtract the exponent of the divisor from that of the dividend in order to obtain that of the quotient.

$$b^{l-l'} = \frac{n}{n'}$$

Since $l - l'$ is the exponent of the power to which it is necessary to raise b the base, in order to produce $\frac{n}{n'}$ it follows that $l - l'$ is the logarithm of $\frac{n}{n'}$ i. e. *the logarithm of the quotient is equal to the difference between the logarithms of the divisor and dividend.*

EXAMPLE.

Divide 3905073 by 2421
The logarithm of 3905073 is 6·59162
" " " 2421 is 3·38399
The logarithm of $\frac{3905073}{2421}$ = 1613 is 3·20763 diff. of logs.

---

[*] As the number 3905073 is too large to be found in the tables, the method of finding its logarithm from the tables must be postponed till the explanation of such cases, further in advance.

Before explaining other operations by means of logarithms, we shall exhibit some principles derived from those just demonstrated.

50. The base of the common system being 10, the common logarithm of 10 is 1. (Art. 44.) Hence if any number be multiplied or divided by any number of times 10, the logarithm of the result will be equal to the logarithm of the given number increased or diminished by the same number of times 1. This 1 being an entire number, the decimal part of the logarithm of the given number will not be altered by this addition or diminution, but only the characteristic.

Thus 39794, which is the decimal part of the logarithm of 2500, is also of 25000, and of 250000, or of 250, or of 25.

The characteristics belonging to these different numbers are different. That of the log. of 2500 is 3; that of the log. of 25000 is 4; that of the log of 25 is 1. (See Art. 47.)

Any number is divided by a multiple of 10, by pointing off from the right as many places for decimals, as the divisor is times 10.

Thus 2348 divided by 10, by 10 twice, by 10 three times, becomes successively 234·8, 23·48, 2·348. The decimal part of the logarithms of these last three numbers, will be the same, viz. 37070, the characteristic being one less each time that we divide by 10 or remove the decimal point one place to the left. Because to divide by 10 it is necessary (see the last Art.) to subtract 1, which is the log. of 10, from the log. of the dividend. The characteristic of the first, 234·8, which is between $100 = 10^2$ and $1000 = 10^3$, is 2. The characteristic of the second is 1; and the characteristic of the last is 0, since 2·348 is less than 10, or 10^1.

The decimal part of the logarithm of a number consisting of significant figures, either followed or preceded by ciphers, will be the same as if the ciphers were absent. Thus the decimal part of the logarithm of 482000 or of ·00482 is the same as the decimal part of the logarithm of 482.

The following table illustrates the theory of the characteristic.

The characteristic of the log. of 482000 is 5
of 482 is 2
of 4·82 is 0
of ·482 is -1
of ·0482 is -2
of ·00482 is -3

From the above, it appears that the characteristic of the logarithm of a decimal fraction is negative; the decimal part of the same logarithm is, however, positive. The actual value of the whole logarithm will be therefore a negative quantity somewhat less than the characteristic. That the logarithms of proper fractions ought to be negative, appears from the fact, that since a fraction expresses the quotient of the numerator divided by the denominator, applying the rule for division by logarithms, the greater logarithm would have to be subtracted from the lesser, and the remainder would of course be negative.

From the above principles are derived the following rules:

1. To find the logarithm of a number consisting of significant figures with any number of ciphers annexed, *find the logarithm of the significant figures, and make the characteristic one less than the number of figures in the given number including the ciphers.*

2. To find the logarithm of a decimal or mixed number, *consider the number as entire; find the decimal part of its logarithm, and make the characteristic one less than the number of figures in the entire part of the given number.*

3. To find the logarithm of a decimal number having ciphers at the left; *lock for the logarithm of the significant figures, and make the characteristic negative* and one more than the number of ciphers at the left of the given decimal.*

EXAMPLES.

The logarithm of 3266000 is 6·51402
of 114·1 is 2·05729
of ·001684 is $\bar{3}$·22634

51. Case II.—We proceed now to the method of determining the logarithm of a number beyond the limits of the table. This method is by a simple calculation from the logarithms of numbers which the table contains, and depends upon the fact that the difference of any two numbers bears the same proportion to the difference of their logarithms, that the difference of two other numbers does to the difference of *their* logarithms, which is nearly true. (See Algebra, p. 284, note.)

Take two numbers in the table differing from each other by 100, as the numbers 843700 and 843800, and a third number 843742 differing from the first of these by 42. The logarithm of the first number 843700, being the same as that of 8437, is given by the tables, and
is 5.92619
The logarithm of the second number 843800 is 5·92624

Their difference is 5

---

* It is customary to write the negative sign over the characteristic, thus, $\bar{2}$·1756348. It affects the characteristic alone and not the decimal part of the logarithm, which must be considered as +.

which may be found by subtraction, but to save this trouble the subtraction is performed, and the difference is written in the margin (in the tables of Callet in the right hand column marked dif.) Then on the principle that the difference of numbers is proportional to the difference of their logarithms, we have

$$\begin{array}{ccccccc} \text{diff. of numbers.} & & \text{diff. of logs.} & & \text{diff. of num.} & & \text{diff. of logs.} \\ 100 & : & \cdot00005 & :: & 42 & : & x \end{array}$$

hence,

$$x = \frac{\cdot00005 \times 42}{100} = \cdot0000210$$

adding this to the logarithm of 843700 which is 5·92619
·0000210

the sum, rejecting the two last places 10 which
go beyond the usual number is 5·92621
which is the logarithm of 843742.

Had the first two numbers differed by 1000 instead of 100 the divisor in the value of x would have been 1000, and the quotient would have extended three places beyond the usual. Had they differed by 10, the quotient would have extended one place beyond.

The inaccuracy of this method increases with the number of additional figures beyond four, in the number the logarithm of which is to be found.*

From the above process may be observed the following rule:

To find the logarithm of a number beyond the limits of the table. Enter the table with the first four figures of the given number, and find the corresponding logarithm. From the right hand margin take out the difference between this logarithm and the next in order in the table, and multiply it by the remaining figures of the proposed number, reject from the product as many figures to the right as there are in the multiplier, and add the rest of the product to the logarithm already found.

EXAMPLE.

1. Required the logarithm of 739245.

The decimal part of the log. of 7392 is 86876.

---

* The same process applies to most other tables as well as tables of logarithms. In all tables the column corresponding to the column of numbers here is called the column of *arguments*. The other columns contain functions depending on these arguments.

$$\begin{array}{lr}
\text{The number in the margin is} & 6 \\
\text{Multiplying this by the remaining figures of the given number} & 45 \\
\hline
\text{Product,} & 270
\end{array}$$

From this product reject as many figures to the right as are contained in the multiplier, that is two in this case, and add the rest to the logarithm before found, namely, 86876

The sum is 86879*

which is the decimal part of the log of 739245 required. Prefixing the proper characteristic, we have 5·86879.

EXAMPLE II.

Required the log. of 8193217

$$\begin{array}{rcr}
\text{log. of } 8193 = 91344 & \qquad & 217 \\
1 & & \text{dif.} = \;\; 5 \\
\hline
\text{log. of } 8193217 = 6.91345 & & 1\cdot085
\end{array}$$

To facilitate the above calculations several smaller tables will be found in the right hand margin of the page, the use of which may be explained from an example. Suppose it be required to find the logarithm of 276738. After cutting off 38 on the right, I find the logarithm of the first four figures, 2767, to be 44201. The difference between this logarithm and the next is 16. In the left hand column of the little table in the margin headed 16, I find the first of the two figures cut off, 3, and against it the number 5, which I place under the logarithm already found, as seen in the scheme below. Again I find in the same way the second of the two figures cut off, 8, and against it 13, which I place as in the scheme below one place to the right. Adding these numbers taken from the little table to the first logarithm found, the sum 44207 is the logarithm sought.

$$\begin{array}{r}
44201 \\
5 \\
13 \\
\hline
44207
\end{array}\text{ or with the characteristic } 5\cdot44207$$

The right hand figure 3 of the 13, which would carry the decimal places beyond five, is rejected. If it were any figure greater than 5, we should add 1 to the last figure of the result, 7.

The following example is worked with the tables of Callet in the same manner.

---

* We add 1 for the $\frac{70}{100}$ rejected which is more than $\frac{1}{2}$.

Find the logarithm of 8193217.

$$\begin{array}{rr} \text{log. of } 81932 = & 9134536 \\ & 5 \\ & 37 \\ \hline & 6{\cdot}9134545 \end{array}$$

The 7 neglected in adding up the above numbers being more than 5, or ½ of the preceding place, 1 is added to that place.

For the theory of the above see Algebra, at the end of Art. 214.

PROBLEM II.

To find the number corresponding to any given logarithm.

52. By referring to the proportion of Art. 51, and putting the value of x for the fourth term, we have

diff. of num.		diff. of logs.		diff. of num.		diff. of logs.
100	:	00005	::	42	:	000021

Instead of the 42 being given and the 000021 required as before, the 000021 is now given and the 42 required.

The first term of the proportion is 100 or 1000, &c., and the second term is diff. in the margin, to find the third term multiply the extremes and divide by the second term

$$42 = \frac{000021 \times 100}{00005}$$

Hence the following

Rule.—*To find the number corresponding to any given logarithm.*

Seek for the decimal part of the given logarithm. If we find a logarithm exactly agreeing with that given, then the number, which the table shows us to belong to the logarithm found, will be the required number. If, however, as is most likely, we do not find the proposed logarithm exactly, then we are to take out the number corresponding to the next less logarithm; this number will of course fall short of that required, but the deficiency may be supplied as follows. Subtract the tabular logarithm from the given one, annex ciphers to the remainder at pleasure, and divide it by the diff. number in the margin, and annex the quotient to the number already taken from the table.

N. B. Should there be a quotient figure without annexing a cipher to the dividend, this quotient figure must be added to the last figure of the number taken from the table. Should it be necessary to annex two ciphers before obtaining a quotient figure, a cipher must be placed in the quotient, and annexed with the figures that come after, to the number taken from the table.

The logarithm next greater than that given may be taken from the tables, and the latter subtracted from the former, in which case you would subtract the quotient obtained by dividing the difference as above, instead of adding it.

EXAMPLES.

1. Find the number the log. of which is 5·86879
The decimal part of the next less log. is that of 7392 = ·86876
Their difference is 3

Annex ciphers to this diff. and divide by the diff. number in the margin, which is 6.

$$6\overline{)300}$$
$$50$$

Annex the quotient 50 to the number 7392 before found, and you have the number required corresponding to the given logarithm, namely, 739250. This number contains six figures, one more than the characteristic of the given logarithm. In every case a sufficient number of ciphers must be annexed to obtain quotient figures enough, when appended, to make the whole of the number which thus results contain one more figure at least than is expressed by the characteristic of the given logarithm. If more quotient figures still be obtained, they will occupy the place of decimals.

2. Find the number of which the log. is 2·91345
Next less log. that of 8193 = 91344
Number required is 819·32 5)10
2

53. To find the number corresponding to any given logarithm by the tables of Callet, seek the nearest logarithm in the tables, and subtract it from the given as directed above, then seek the remainder in the right hand column of the little table nearest, and if it be found, or a number not differing more than unity from it, the figure on the left of this number will express the sixth figure of the number required.

EXAMPLE.

2·5386717
The nearest log. is ·5386617 its number is 34567
Subtract and 100 is the remainder.

The nearest number in the right hand column of the little table adjoining is 101, against which on the left is the figure 8, and the number sought is 345·678

If the diff. (100 above) is not found exactly in the little table, take the nearest number to it, and take the difference between these again, and annex to it a cipher, seek this result again in the right hand column of the little table, the figures on the left of the two numbers taken out of the right hand column of the little table will be the sixth and seventh figures of the number required A third remainder might be found in the same manner, and an eighth figure of the required number be found.

EXAMPLE.

0·4971499
The nearest log. is 4971371
Diff. 128
nearest to which in the little table is 124 number on its left is 9.
diff. with cipher annexed 40
nearest to this in the little table is 41 number on its left is 3.

The last two figures of the number sought are 93, and the number itself is 3·141593.

The same method is applicable to our tables, though not with the same degree of accuracy.

EXAMPLE.

4·90835
Nearest log. 90832 corresponding number 8097
Difference 3 on its left in the marginal table is 5.

The number required is 80975.

EXAMPLE.

0·06180
Nearest log. 06145 corresponding No. 1152.
Diff. 45

The nearest No. to which in the little table is 34 which subtract; on its left is 9.

The rem is. 11 on left of which in the marginal table is 3.

The number required is therefore 1·15293.

EXAMPLES IN MULTIPLICATION AND DIVISION BY LOGARITHMS.

54. 1. Required the product of 26784 and 7·865.

log. of 26784* is 4·42787
log. of 7·865 is 0·89570
Their sum is 5·32357

5·323574 is the log. of 210656, which last number is, therefore, the product required.

---

* In looking for the log. of this number, look first for that of 2678, multiply the tab. diff. by 4, the last figure of the given number, and cut off one figure from the product.

2. Required the product of 3·586, 2·1046, ·8372, and ·0294.

$$\begin{array}{lrcr}
\text{log. of} & 3{\cdot}586 & \text{is} & 0{\cdot}55461 \\
\text{of} & 2{\cdot}1046 & \text{is} & 0{\cdot}32317 \\
\text{of} & {\cdot}8372 & \text{is} & \bar{1}{\cdot}92283 \\
\text{of} & {\cdot}0294 & \text{is} & \bar{2}{\cdot}46835 \\
\hline
\text{Product} & {\cdot}18576 & & \bar{1}{\cdot}26896
\end{array}$$

Instead of using negative characteristics, a method is sometimes employed of taking the difference between the negative characteristic and 10, which is really adding 10 to the negative characteristic, and writing this difference as a positive characteristic; thus, in the above example,

$$\begin{array}{rlr}
3{\cdot}586 & \text{log.} & 0{\cdot}55461 \\
2{\cdot}1046 & \text{log.} & 0{\cdot}32317 \\
{\cdot}8372 & \text{log.} & 9{\cdot}92283 \\
{\cdot}0294 & \text{log.} & 9{\cdot}46835 \\
\hline
{\cdot}185764 & \text{log.} & 9{\cdot}26896
\end{array}$$

twice 10 must be rejected from the sum. That is a 10 for each positive characteristic employed in the place of a negative.

The result thus obtained $\bar{1}$·26896 is written 9·26896, 10 being added again to avoid the negative characteristic.

3. Divide 28·654 by 127·34.

$$\begin{array}{lrcr}
\text{log. of} & 28{\cdot}654 & \text{is} & 1{\cdot}45718 \\
\text{of} & 127{\cdot}34 & \text{is} & 2{\cdot}10496 \\
\hline
& & \text{difference} & \bar{1}{\cdot}35222
\end{array}$$

$\bar{1}$·35222 is, therefore, the log. of the quotient which from the tables, observing the converse rule for pointing off decimals according to the characteristic (3 Art. 50), is ·225019.

Divide ·06314 by ·007241.

$$\begin{array}{lrcr}
\text{log. of} & {\cdot}06314 & \text{is} & \bar{2}{\cdot}80030 \\
\text{of} & {\cdot}007241 & \text{is} & \bar{3}{\cdot}85980 \\
\hline
\text{Quotient} & 8{\cdot}7197 & & 0{\cdot}94050
\end{array}$$

55. We shall now demonstrate rules for raising numbers to powers, and for extracting the roots of numbers, by means of logarithms.

Resume the equation,

$$b^l = n$$

raising both members to the m^{th} power, we have, observing the rule of Algebra, which is to multiply the exponent by the degree of the power,

$$b^{lm} = n^m$$

From this last equation, it appears that lm is the power to which it is necessary to raise the base b in order to produce n^m; hence the following

RULE.—To raise a number to any power, by means of logarithms, *multiply the logarithm of the given number by the exponent of the power, and the product will be the logarithm of the power.*

EXAMPLES.

1. Required the 4th power of ·09163

$$\begin{array}{lr} \text{log. of } {\cdot}09163 \text{ is} & \bar{2}{\cdot}96204 \\ \text{Multiply by} & 4 \\ \hline \text{Product} & \bar{5}{\cdot}84816 \end{array}$$

$\bar{5}{\cdot}84816$ is the log of ·000070494, which last number is the power required.

2. Required the tenth power of ·64.

$$\begin{array}{lr} \text{log. of } {\cdot}64 & \bar{1}{\cdot}80618 \\ & 10 \\ \hline \text{Power } {\cdot}0115293 & \bar{2}{\cdot}06180 \end{array}$$

In multiplying the first decimal place by 10, the product is 80, then 10 times $\bar{1}$ is $\overline{10}$, and 8 to carry is $\bar{2}$.

The same example, with positive characteristics, according to the method pointed out on p. 45 would stand thus

$$\begin{array}{r} 9{\cdot}80618 \\ 10 \\ \hline 8{\cdot}06180 \end{array}$$

The 10 added to the $\bar{1}$ is repeated 10 times and therefore 100 must be rejected from the product, which leaves $\bar{2}$ for the characteristic to be written 8. The rule for placing the decimal point in the number corresponding to the logarithm will be to place one less cipher after the decimal point than is expressed by the difference between the characteristic and 10.

56. To find a rule for extracting the root of a number by means of logarithms, assume again the equation

$$b^l = n$$

Take the m^{th} root of both members, applying in the first member the rule to divide the exponent by the number expressing the degree of the root, and there results

$$b^{\frac{l}{m}} = \sqrt[m]{n}$$

$\frac{l}{m}$ is here plainly the logarithm of $\sqrt[m]{n}$; hence the following

Rule.—To extract the root of a number by means of logarithms, *divide the logarithm of the given number, by the index of the root, and the quotient will be the logarithm of the root.*

EXAMPLES.

1. Required the 4th root of ·434296.

log. of ·434296 $\bar{1}$·63779

$\frac{1}{4}$ of this logarithm is obtained by observing that the index, which alone is negative, must be divided separately, as we should divide a minus term, followed by a plus term in Algebra; the $\bar{1}$ can be rendered divisible by borrowing $\bar{3}$, and afterwards carrying + 3 before the 6, rendering it 36; that is, the proposed logarithm is viewed under the form $\bar{4}$ + 3·63779.

The quotient is $\bar{1}$·90945* which is the logarithm of ·8118, the fourth root required.

2. Required the 10th root of 2.

log. of 2 0·30103

Divide this by 10 0·03010 quotient.

which is the log. of 1·07177, the root required.

3. Required the cube root of ·00048.

log. of ·00048 $\bar{4}$·68124

$\frac{1}{3}$ of it $\bar{2}$·89375 = log. of ·078297, the root.

TABLES OF LOGARITHMIC SINES, TANGENTS, &C.

57. This is Table XXVII. It contains the logarithm of the sine, tangent, cosine and cotangent, secant and cosecant, corresponding to every degree and minute in the quadrant.†

These logarithms are those of the trigonometrical lines in a circle, the radius of which is 10000000000, or the tenth power of 10, the common logarithm of which is 10. As the sine is never greater than radius, its logarithm will always be less than 10, except for the arc 90°, the logarithmic sine of which is equal to 10.

---

* The last quotient figure is nearer 5 than 4.

† Without this table we should have been obliged to employ the other two tables which have been already described, as follows. First we must have found the natural sine, tangent, &c., of the given arc or angle in Table XXIV. then with this have entered Table XXVI. and found its logarithm. The tables of Callet do not contain the columns of secants and cosecants.

PROBLEM.

58. To find from the table the logarithm of the sine, tangent, or cosine of the number expressing any arc.

Case I. If the given number be composed of degrees and minutes, seek first for the number of degrees among those which are written at the top or bottom of the pages; at the top and on the left of the page if it be less than 45°; at the bottom and on the right of the page if it be greater. Run the eye down the first column which goes on increasing from top to bottom, if the number of degrees is found at the top of the page; or up the last column, which goes on increasing from the bottom upwards, if the number of degrees is found at the bottom; run the eye, I say, through one or the other of these columns in the direction in which it increases until you have found the number of minutes given; upon the same horizontal line with the minutes thus found you will find the logarithm of the sine, cosine, tangent, or cotangent which you seek. In order not to mistake the column, it is necessary to consult the title at the head of the column, if the number of degrees given is at the top of the page, but if it is at the bottom, the inferior title must be consulted.*

EXAMPLES.

1. Required the logarithmic sine, tangent, secant, cosine, cotangent, and cosecant of 19° 55′.

I find 19° at the top of page 204, I descend the first column at the left marked M, which goes on increasing downwards till I find 55′; upon the same horizontal line, and in the column entitled sine at top, I find 9·53231, in the column entitled cosine 9·97322, in the column of tangents 9·55910, and in that of cotangents 10·44090, that of secants 10·02678, that of cosecants 10·46769; and these numbers are therefore the numbers required.

2. Required the logarithmic sine and tangent of 70° 10′.

I find 70° at the bottom of p. 204†; I ascend the last column marked M at bottom which goes on increasing upwards; I find 10′ in that column; upon the same horizontal line I find in the column marked sine at bottom 9.97344, and in the column marked tangent at bottom 10·44288, which are the logarithms sought.

---

* The columns marked Hour A. M. and P. M. are connected with Nautical and Practical Astronomy, and will be explained under the proper head.

† For the reason mentioned in note p. 33, the paging of the tables will be found to exhibit gaps.

3. Required the logarithmic sine of 159° 20′.

This will be the same with that of its supplement 20° 40′. For convenience the supplements of arcs in the 1st quadrant are placed on the right of the page in the tables at top, and on the left at bottom. By looking therefore for 159° at top on the right, and immediately under for 20′ in the column M, on the range with this in the column entitled sine at top I find 9·54769, the same that would have been found for 20° 40′.

59. Case II. If the given number be composed of degrees, minutes, and seconds, find the logarithm of the degrees and minutes as above, and then to know how much this should be increased for the given number of seconds, in case of the sine or tangent, or diminished in case of the cosine or cotangent, observe that the number in the column marked Diff. is the increase of the logarithm for the number of seconds in the column headed M, against which it stands, and will be the quantity to add to the logarithmic sine or tangent before found, or to subtract from the logarithmic cosine or cotangent.

The number in the column Diff. is calculated by subtracting one of two consecutive logarithms in the table, which differ by 1′, from the other,* and dividing the remainder by 60, the number of seconds in a minute; the quotient is the difference of logarithms corresponding to a difference of 1″ in the numbers to which they belong, or is the increase of the logarithm for 1″ increase of arc. This quotient, multiplied by any number of seconds, will give the increase of the logarithm for that number of seconds. This calculation depends upon the principle mentioned at Art. 51, that the differences of logarithms are proportional to the differences of their corresponding numbers. See also Art. 23 App. I.

60. The logarithmic sines and cosecants, cosines and secants, tangents and cotangents, have each pair but one column of differences between them, the reason of which will appear from the following demonstration.

The secant and cosecant may be easily computed from the cosines and sines. Thus (Art. 33):

$$\text{sec.} = \frac{\text{R}^2}{\cos}$$

hence,

$$\text{log. sec} = 20 - \text{log. cos}$$

for log. of the quotient = difference of logs. (Art. 49); and log. of the square of a number equal to twice the log. of the number (Art. 56), and log of R = 10.

---

* This diff. is given in the first five pages of the table so that the seconds must be calculated as here described when the given arc is 4° or less, and any number of minutes.

To obtain the log. secant, therefore, we have this

RULE.—*Subtract log. cosine from* 20.

Also (Art. 35),

$$\text{cosec} = \frac{\text{R}^2}{\sin}$$

hence,

$$\log.\ \text{cosec} = 20 - \log.\ \text{sine}$$

whence this

RULE.—*To obtain log. cosec, subtract log. sine from* 20.

From the above it appears that as the log. cos decreases with the increase of arc the log. sec increases by the same amount, and as the log. sin increases the log. cosec decreases by the same amount.

Again, by Art. 37, we have

$$\tan \times \cot = \text{R}^2$$

applying logarithms to this equation, since the log. of a product = the sum of the logs. of the factors, and the log. of a power = log. of the number raised to the power multiplied by the index of the power, we have

$$\log.\ \tan + \log.\ \cot = 2 \log \text{R} = 20$$

log. R being 10. Therefore having two arcs a and b, since log. tan + log. cot in both is 20 we have

$$\log.\ \tan a + \log.\ \cot a = \log.\ \tan b + \log.\ \cot b,$$ or transposing,

$$\log.\ \tan a \sim \log.\ \tan b = \log.\ \cot a \sim \log.\ \cot b,$$

that is, *the difference of the logarithmic tangents of two arcs is equal to the difference of their logarithmic cotangents.*

EXAMPLES.

1. Required the logarithmic sine of 40° 26′ 28″.

I find the log. sine of 40° 26′ to be 9·81195; in the column M, at the left, I find the given number of seconds 28, and on the same horizontal line in the column of diff. I find 7, which added to the log. before found

$$\begin{array}{rr} & 9{\cdot}81195 \\ & 7 \\ \hline \text{gives} & 9{\cdot}81202^* \end{array}$$

The logarithmic tangent of any given number of degrees, minutes, and seconds, is found in a similar manner from the column entitled tangent.

---

[*] A similar arrangement for finding the difference corresponding to any number of seconds will be found in the table of natural sines and cosines, for the former on the left and for the latter on the right of the page.

2. Required the logarithmic cosine of 8° 40′ 40″.

I find the cosine of 8° 40′ to be 9·99501; the tabular difference in the adjoining column against the seconds 40″ is 1; subtracting* this result from 9·99501, the remainder is 9·99500, the logarithmic cosine sought.

The difference for the seconds may be calculated as follows.

3. Required the log. sine of 32° 10′ 23″.

32° 10′	log. sin	9·72622
32° 11′	" "	9·72643
	Diff.	21
		23
		63
		42
		60)483
	Diff. for 23″	8
32° 10′ log. sin		9·72622
32° 10′ 23″ log. sin		9·72630

61. In the tables of Callet are found the logarithmic sines, tangents, cosines, and cotangents for every 10″ in the quadrant; and the columns of differences contain the differences of the consecutive logarithms, or the increase of the logarithm for 10″ at that part of the quadrant. To take out therefore a logarithmic sine, &c., from these tables, take out for the degrees, minutes, and tens of seconds, and take out also the number from the column of diff.; cut off one figure on the right of the latter, which is equivalent to dividing by 10, and multiply it by the number of seconds by which the given arc differs from an exact number of tens of seconds. The product will be the number by which to increase or diminish the logarithm already taken out, according as the trigonometrical line to which it corresponds is an increasing or decreasing function of the arc.

EXAMPLES OF THE APPLICATION OF THE TABLES OF CALLET.

1. To find the log. tan of 49° 12′ 25″·8

Of 49° 12′ 20″ log. tan is	10·0639854†
Tab. diff. 425 × 5·8	246·5
	10·0640100 log. required.

---

* It will be recollected that as the arc increases in the first quadrant the cosine diminishes.

† The characteristic in the column of tangents and cotangents when 10, is printed 0. and when 11, is printed 1., in the tables of Callet.

2. To find the log. cot of $101^\circ\ 25'\ 43'' =$ log. cot of $78^\circ\ 34'\ 17''$

Of $78^\circ\ 34'\ 10''$ log. cot is 9·3057605

Diff. for $10''$ or 1084×7 758·8

9·3056846 log. required.

In the tables of Callet are to be found the logarithms of trigonometrical lines of arcs given in grades, &c., of the centesimal division of the circle. The following is an example of the use of the table.

To find the log. sine of $92^{g}\ 75^{\backprime}\ 84^{\backprime\backprime}$

of $92^{g}\ 75^{\backprime}$ the log. sin is 9·9971776

tab. diff. $78 \times 84 =$ 65·52

log. required is 9·9971842

The first 4 figures of the log. 9·997 are taken from the top or bottom of the column, the former when the logarithm is above a black horizontal line drawn across the column, the latter when below.

As the diff. 78 is the diff. of two consecutive logarithms corresponding to $100^{\backprime\backprime}$ two figures must be pointed off to the right after multiplying by the $84.^{\backprime\backprime}$

PROBLEM.

62. *To find the degrees, minutes, and seconds answering to any given logarithmic sine, cosine, tangent, or cotangent.*

The method is, of course, exactly the reverse of that just given. Look for the given logarithm in the proper column, which you will know from its title, either at the top or bottom, and if you find it exactly, the degrees will be found at the top of the page, and the minutes on the same horizontal line with your logarithm, in the first column at the left, if the title of the column be at top, but the degrees will be found at the bottom of the page, and the minutes in the column at the right, if the title of the column which contains your logarithm be at the bottom.

If the given logarithm cannot be found, take the next less logarithm contained in the tables, subtract it from the given, and seek the remainder in the column marked diff.; the number on the same horizontal line in the column M is seconds, which add to the degrees and minutes belonging to the logarithm found in the tables, if your given logarithm be that of a sine or tangent, but which subtract from the degrees and minutes, if a cosine or cotangent.

Or more accurately, to find the seconds multiply the remainder above mentioned, by 60, and divide the product by the difference between two consecutive logarithms in the table.*

EXAMPLES.

1. Required the number of degrees, minutes, and seconds, of which the logarithmic sine is 9•88005.

I find the next less logarithm in the column marked sine at bottom, to be 9•87996, which subtracted from the given logarithm, leaves 9; this found in the column diff. adjoining, against it in the column M is 50, which is seconds. Taking the degrees from the bottom of the page, and the minutes from the column at the right, and in the same horizontal line with the logarithm 9•87996, I have 49° 20′ 50″ for the number required.

Or more accurately, since the diff. 9 corresponds to any number of seconds from 48″ to 52″ calculate as follows:

Given log.	9•88005		9•87996
Nearest log.	9•87996	consec log.	9•88007
Diff.	9	Diff.	11

9
60
11)540
49

2. Required the number of degrees, minutes, and seconds, of which the log. cotangent is 10•00869.

I find the next less logarithm in the table to be 10•00859, that of 44° 26′, which subtracted from the given logarithm, leaves 10, corresponding to which in the column M is either 23″ or 24″; more accurately

10
60
25)600(24″
50
100
100

and the required number is 44° 26′—24″ or 44° 25′ 36″.

---

* This last rule is on the principle that the difference of the logarithmic functions is proportional to the difference of their arguments, the difference of the arguments in this case being 60″.

3. Required the log. sec of 48° 35′ 27″.

BY TABLE XXVII.

48° 35′ log. sec 10·17945
Diff. for 27″ 6
48° 35′ 27″ log. sec 10·17951

The tables of Callet not containing the logarithmic secants and cosecants, the calculation of this example from his tables would be as follows, by the rule at Art. 60.

log. cos 48° 35′ 27″ = 9·82049
log. sec 48° 35′ 27″ = (20 — 9·82049) = 10·17951

4. Required log. cosec 35° 27′ 24″. Ans. by Tab. XXVII., 10·23651
or log. sin 35° 27′ 24″ = 9·76349
log. cosec 35° 27′ 24″ = 20–9·76349 = 10·23651

A method of finding with greater accuracy the sine and tangent of a very small arc, or the cosine and cotangent of one near 90°, is pointed out at Art. 8 App. I.

To find the trigonometrical lines of arcs greater than 90°, observe the rule at Art. 17.

SOLUTION OF RIGHT ANGLED TRIANGLES, WITH THE AID OF LOGARITHMS.

EXAMPLE.

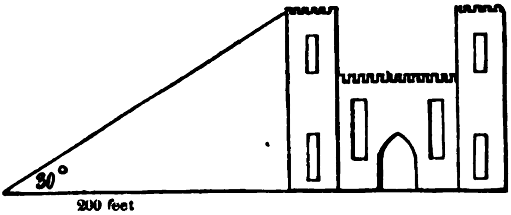
63. Referring to the example of Art. 39, where the hypothenuse

$$a = \frac{10^{10} \times 15}{\sin 35^\circ}$$

employing 10^{10} as R, instead of 1, because the tables which we are about to use are constructed with that value of R, we have, by the rules for multiplication and division of logarithms

log. of 10^{10} = 10·00000
add log. 15 = 1·17609
11·17609
Subtract log. sin 35° = 9·75859
Remainder 1·41750 = log. of 26·15 = the hyp.

EXAMPLE II.



Referring to Art. 41 where the height of the tower c is

$$c = \tan 30° \times 200^{*}$$

which becomes, when the radius 1 in the first term of the proportion is changed into 10^{10}

$$c = \frac{\tan 30° \times 200}{10^{10}}$$

we have, by applying logarithms

$$\begin{array}{rl} \text{log. tan } 30° = & 9{\cdot}76144 \\ \text{add log. } 200 = & \underline{2{\cdot}30103} \\ & 12{\cdot}06247 \\ \text{Subtract log. } 10^{10} = & \underline{10} \\ & 2{\cdot}06247 = \text{log. } 115{\cdot}4 \end{array}$$

64. I. Another definition besides that given at Art. 14 for the sine of an angle, is *the ratio of the hypothenuse to the side opposite the angle* in any right angled triangle of which the angle forms one of the elements.

For from formulas (1) Art. 38 may be obtained.

$$\sin \text{B} = \frac{b}{a}, \text{ and } \sin \text{C} = \frac{c}{a} \qquad (1)$$

II. A corresponding definition for the cosine of an angle is *the ratio of the hypothenuse to the side adjacent the angle* in any right angled triangle of which it forms an element. For, form. (3) Art. 38

$$\cos \text{B} = \frac{c}{a}, \text{ and } \cos \text{C} = \frac{b}{a}^{\dagger} \qquad (2)$$

---

[*] It is evident that radius must be understood in the second member of this expression, because a line c cannot be equal to the rectangle of two lines. (Art. 31.)

[†] The hypothenuse, which is the greater side, is evidently the denominator of the ratio, as the sine and cosine to radius unity are always fractions.

III. In a similar manner the tangent of an angle is *the ratio of the side adjacent to the side opposite the angle* in any right angled triangle of which the angle forms an element. For Art. 41, (3) and (4)

$$\tan B = \frac{b}{c} \text{ and } \tan C = \frac{c}{b} \qquad (3)$$

From these definitions the following consequences flow.

IV. *The hypothenuse multiplied by the sine of one of the acute angles of a right angled triangle will give the side opposite to the angle.*

For from (1) above clearing of fractions we have

$$a \sin B = b \text{ and } a \sin C = c \qquad (4)$$

V. *The hypothenuse multiplied by the cosine of the angle will give the side adjacent.* For from (2) above

$$a \cos B = c, \text{ and } a \cos C = b^* \qquad (5)$$

VI. *The side adjacent multiplied by the tangent* will give the *side opposite.* For from (3) above

$$c \tan B = b, \text{ and } b \tan C = c \qquad (6)$$

VII. To *find* the *hypothenuse* when a *side* and *angle* are *given*, we must still use the sine and cosine of the angle, but as a *divisor ;* the *sine* when the side opposite is given and serves for a dividend, the *cosine* when the side adjacent. For from formulas (4) and (5) above may be obtained

$$a = \frac{b}{\sin B}, \; a = \frac{c}{\sin C}, \; a = \frac{c}{\cos B}, \text{ and } a = \frac{b}{\cos C} \qquad (7)$$

VIII. When one of the *perpendicular* sides is given with an angle to find the other, the *tangent* of the angle is the *multiplier* of the side *adjacent* the angle to *find* the side *opposite*, and is the *divisor* of the side *opposite*, to find the side *adjacent.* See (6) above from which also may be obtained

$$c = \frac{b}{\tan B}, \text{ and } b = \frac{c}{\tan C} \qquad (8)$$

---

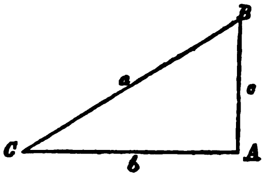
* When the hypothenuse is given with an acute angle it will be necessary always to multiply the hypothenuse by either the sine or cosine of the acute angle to obtain the other side ; by *the sine* when it is the side *opposite* which is required, and by *the cosine* when it is the side *adjacent.* The memory will be aided by observing that for radius unity the sine and cosine are always fractions, and therefore the hypothenuse, which is the larger side, must evidently be *multiplied* by these in order to obtain the other sides.

IX. Radius unity has no effect either as a multiplier or divisor, nevertheless when using the tables in which radius is 10^{10} or 10,000,000,000, it is necessary to know how the radius enters into the products or quotients formed by the above rules. And the following consideration will always show, viz. that two equal quantities must be homogeneous. A line cannot be equal to a surface or the rectangle of two lines. Therefore all the *products* formed by the above rules must be understood to be *divided* by radius. Neither can a line be equal to the ratio or quotient of two lines, for this is an abstract number. Therefore all the *quotients* formed by the above rules must be understood to be *multiplied* by radius.

In the use of logarithms, therefore, for the solution of right angled plane triangles, in every case two logarithms only will have to be employed, and their sum or difference taken according as the rule which applies to the case requires multiplication or division.

When the sum of the two logarithms is taken, 10 must be rejected from the characteristic, which is in effect dividing by radius, and when the difference is taken, 10 must be added to the characteristic of the minuend, which is in effect multiplying by radius. All this will be easily comprehended from the following examples.

EXAMPLE I.



In the right angled triangle ABC right angled at A, given the hypothenuse $a = 493·7$, and the angle $C = 65° 40'$ to find the side c.

By Rule IV. above, observing that addition of logarithms corresponds to multiplication of the corresponding numbers, and rejecting 10 from the sum, according to IX., the calculation will be as follows :

a	493·7	log.	2·69346
C	65° 40′	log. sin	9·95960
c	449·8	log.	2·65306

The sum of the logarithms 2·69346 and 9·95960 which is 12·65306 after rejecting 10 from index will be the logarithm of 449·8 as may be seen by inspecting the tables, and this number is the value of the required side *c*.

EXAMPLE II.

The same things being given to find the side *b*. By Rule V. above

a	493·7	log.	2·69346
c	65° 40′	log. cos	9·61494
b	203·4	log.	2·30840

Both the above operations may be conveniently connected together by preparing first a blank form thus:

a	log.	log.
c	log. sin	log. cos
c	log.	
b		log.

in which the *arguments* or elements of the triangle occupy the first column, the trigonometrical functions of these arguments necessary for the calculation of the side *c* the second, and those necessary for the calculation of the side *b* the third. This form is then to be filled up as follows

a	493·7	log.	2·69346	log.	2·69346
c	65° 40′	log. sin	9·95960	log. cos	9·61494
c	449·8	log.	2·65306		
b	203·4			log.	2·30840

The log. of 493 when found in the tables is written in two columns, and the log. sin and log. cos of 65° 40′ can be taken out at one opening of the tables.

EXAMPLE III.

Let *b* = 73·94, c = 57° 20′, to find *a* by VII., subtracting the logarithms for division, adding 10 to the index of the minuend (see IX.) before subtracting, the calculation will be as follows:

b	73·94	log.	1·86888
c	57° 20′	log. cos	9·73219
a	136·9	log.	2·13669

EXAMPLE IV.

Let *c* = 115·3, c = 57° 20′, to find *a*

c	115·3	log.	2·06183
c	57° 20′	log. sin	9·92522
a	136·9	log.	2·13661

EXAMPLE V.

Given $b = 67\cdot84$, $\textsc{c} = 41^\circ\ 4'\ 36''$, to find c and a. By VII. VIII. and IX.

b	67·84	log.	1·8314858*	log.	1·8314858
C	41° 4′ 36″	log. cos	9·8772740	log. tan	9·9403365
a	89·993		1·9542118		
c	59·132				1·7718223

EXAMPLE VI.

Given $a = 84\cdot9$, $b = 53\cdot4$, to find B, C, and c.
For the rule applicable see I.

a	84·9	log.	1·92891
b	53·4	log.	1·72754
B	38° 58′ 28″	log. sin	9·79863
$a+b$	138·3	log.	2·14082
$a\sim b$	31·5	log.	1·49831
$c^2=a^2\sim b^2$†		log.	3·63913 ÷ 2
c	66·0035	log.	1·81957
$\textsc{c} = 90^\circ - \textsc{b} = 51^\circ\ 01'\ 32''$			

EXERCISES IN THE SOLUTION OF RIGHT ANGLED PLANE TRIANGLES.

1. Given $a = 49\cdot63$, $b = 25\cdot42$, to find c, B, and C.
Ans. $c = 42\cdot124$, $\textsc{b} = 59^\circ\ 11'\ 39''$, $\textsc{c} = 30^\circ\ 48'\ 21''$.

2. " $a = 723\cdot1$, $c = 95\cdot4$, to find b, C, and B.
Ans. $b = 716\cdot75$, $\textsc{c} = 82^\circ\ 25'\ 2''$, $\textsc{b} = 7^\circ\ 34'\ 58''$.

3. " $a = 853$, $\textsc{b} = 49^\circ\ 31'\ 22''$, to find the remaining parts.
Ans. $b = 648\cdot8463$, c 553·725.

4. " $a = 940$, $\textsc{c} = 30^\circ\ 20'\ 10''$, to find the other elements.
Ans. $c = 474\cdot769$, b 811·32.

5. " $b = 25$, $c = 17$, to resolve the triangle.
Ans. C 34° 12′ 27″ a 30·2372, B 55° 47′ 33″.

---

* The logarithms in this example are taken from the tables of Callet, and extend to seven places of decimals.

† The product of $a+b$ and $a-b$ which is obtained by adding this logarithm, is equal to a^2-b^2. (See Alg. Art. 13.)

6. Given $b = 42173$, $C = 40° 20'$.

Ans. $a = 55329$, c 358·91, B 49° 40′.

7. " $b = 328$, $B = 74° 25' 18''·7$.

Ans. a 340·5025, c 91·436.

8. " $c = 8·76213$, $C = 89°$.

Ans. a 87644, b 15294, B 1°.

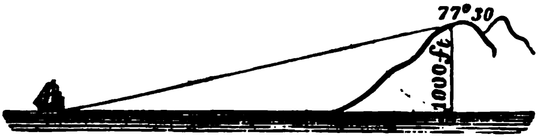
9. " $c = 82·94$, $B = 40° 50' 20''·6$.

Ans. a 109·612, b 71·6914, c 49° 9′ 39″·4.

10. " $b = 81·5$, $c = 92·19$.

Ans. B 41° 28′ 42″, C 48° 31′ 18″, a 123·33.

65. We shall finish the subject of right angled triangles by presenting a case of their practical application which is likely often to occur.



At the top of a mountain whose height was known by the barometer or otherwise to be 1000 feet above the level of the sea, a ship was observed through the tube of the instrument described at Art. 10, and the number of degrees between the tube and a plumb line from the centre of the circle was found to be 77° 30′; required the distance of the ship.

A right angled triangle is here formed, in which are given the perpendicular and angle at the vertex, and the base is required.

Referring to Rule VI.

$$\tan 77° 30' \times 1000 = \text{dist. required.}$$

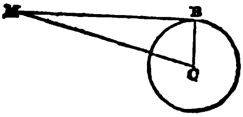
$$\begin{array}{lll} 1000 & \text{log.} & 3·00000 \\ 77° 30' & \text{log. tan} & 10·65424 \\ \hline & & 3·65424 = \text{log. of } 4510·71 \text{ feet.} \end{array}$$

Therefore the distance of the ship is 4510·71 feet, or a little over ⅚ of a mile.

EXERCISES.

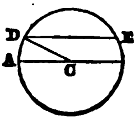
An upright post being 90 feet high, and its horizontal shadow 117 feet, to find the altitude of the sun. *Ans.* 37° 34′ 7″.

The alt. of the sun being 48° 10′ and the length of a horizontal shadow = 201, to find the height of the tower which casts it. *Ans.* 224·54.



To find the horizontal parallax of the moon (angle M), its distance MC being 240,000 miles, and the rad. of the earth BC 3956.

Ans. 56′ 40″.



The lat. of a place D being given, DCA = 38° 14′ and the radius DC of the earth 3956 miles, to find the radius of the parallel DE.

Ans. 3107·43.

To find the side of a regular inscribed figure of 13 sides, when the radius of the circle is 6·7.

Ans. 3·206.

The side of an inscribed regular heptagon being 5·73, find the rad. of the circumscribing circle. *Ans.* 6·6032.

66. It is customary where the subtraction of logarithms, corresponding to the division of numbers, is to be performed, to change this operation into addition by means of what is called the arithmetical complement of each subtractive logarithm. The arithmetical complement of a logarithm is the remainder after taking the logarithm from 10; thus the arithmetical complement of the logarithm 2·32245, is 7·67755.

$$\begin{array}{r} 10\cdot00000 \\ 2\cdot32245 \\ \hline 7\cdot67755 \end{array}$$

It may be formed most conveniently, instead of beginning at the right and subtracting each figure from 10 and carrying one each time throughout, by beginning at the left and subtracting each figure from 9 till you come to the last figure, which subtract from 10.

When we have a logarithm to subtract, we shall obtain the same result by adding its arithmetical complement, and afterwards subtracting 10. Which may be proved as follows:

By the definition arith. comp. log. $b = 10 - \log. b$.

Now add this arith. comp. to some other log. as log. a, the result will be

$$\log. a + 10 - \log. b$$

subtract 10 and there remains

$$\log. a - \log. b$$

The same result as would have been obtained by subtracting log. b from log. a.

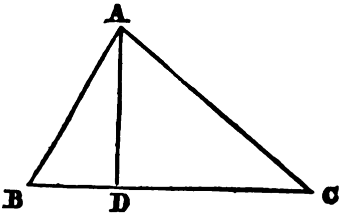
Hence to perform operations containing a number of multiplications and divisions, by means of logarithms, we have the following

RULE.—Write the logarithms of all the multipliers, and the arith. complements of those of the divisors in a column. Add up the whole, and reject as many times 10 from the characteristic of the sum as there have been arith. complements employed.

"This use of the arithmetical complement is not expedient in the solution of right angled triangles, but in the solution of oblique angled triangles it always saves one line of numerical work."

SOLUTION OF PLANE TRIANGLES IN GENERAL.

67. Let ABC be any triangle. From the vertex of one of the angles A, let fall upon the side opposite, the perpendicular AD; the given triangle will be divided into two right angled triangles ABD and ACD.



In the first of these (Art. 64, IV.) we have

$$AD = AB \times \sin B$$

Again, in the right angled triangle ACD, we have

$$AD = AC \times \sin C$$

The first members of the two equations above being the same, the second members are equal, hence

$$\sin B \times AB = \sin C \times AC$$

Turning this equation into a proportion, by making the first product the extremes, and the second the means, we have

$$\sin B : \sin C :: AC : AB$$

That is, *the sines of the angles of any plane triangle, bear the same proportion to each other as the opposite sides.*[*]

From the nature of the above demonstration the selection of the vertex from which to let fall the perpendicular being entirely unrestricted, it is plain that this rule applies to all the angles and sides alike.

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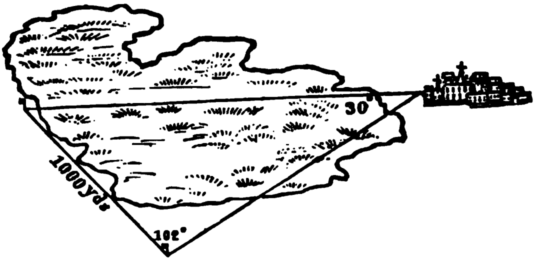
[*] The same demonstration will apply when the perpendicular from the vertex of one of the angles falls upon the side opposite produced.

When therefore two of the three given elements of a triangle, are a side and its opposite angle, the element opposite to the third given element may be found by the proportion which has just been established.

We shall according to our custom suppose a practical problem which introduces the case of solution in question.



Let it be required to ascertain accurately the distance from the town in the figure, across an impassable marsh, for the purpose of estimating the expense of a causeway.

Plant two staves with small flags, the one at the border of the marsh opposite the town where the causeway is to terminate, and the other at some convenient point from which the first and the town may both be seen. Measure the distance between the staves, and let it be 1000 yards; observe also the angle at the second staff by turning the tin tube of the instrument for taking angles first to the staff on the border of the marsh and then to the town; let this angle be 102°; observe also the angle at the town subtended by the line of the two staves, and let this last be 30°. A triangle will be formed in which are known an angle 30°, the side opposite 1000 yards, and another angle 102°, the side opposite to which is required. Then applying the above proportion

sin 30° : sin 102° : : 1000 : length of causeway required.

Arith. comp. log. sin. 30° = 0·30103
(Art. 15) log. sin 102° = log. sin 78° = 9·99040
log. 1000 = 3·00000
3·29143 = log. of 1956

The length of the causeway must be 1956 yards.

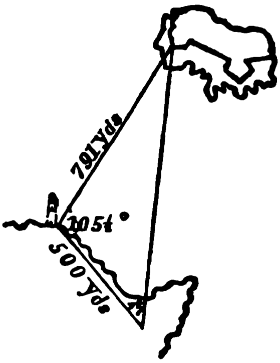
EXERCISE.

In an oblique angled triangle given

$A = 30° \; 20' \; 10''$, $B = 40° \; 10' \; 30''$, $a = 9754$, to find C, b, and c.

Ans. $C = 109° \; 29' \; 20''$, $b = 12458·75$, $c = 18205·71$.

67. This proportion is also applicable to the case where two angles and the interjacent side of a triangle are given. Such is the case in the problem at Art. 10.



One of the angles of that triangle being given equal to 47°, and the other equal to 105° 30′, it will be easy to find the third angle, by recollecting that the sum of the three angles of a triangle is equal to two right angles (Geom. Theorem 15), or 180°; therefore subtracting the sum of the two given, $47° + 105° 30′ = 152° 30$ from 180°, the remainder 27° 30′ is the third angle of the triangle, and is opposite the given side 500 yards. Thus by the proportion

sin 27° 30′ : sin 47° : : 500 : dist. from lt. house to fort.

$$\begin{array}{lrl} \text{arith. comp. log. sin } 27° \ 30' & = & 0{\cdot}33559 \\ \text{log. sin } 47° & = & 9{\cdot}86413 \\ \text{log. } \ 500 & = & \underline{2{\cdot}69897} \\ & & 2{\cdot}89869 = \text{log. of } 791{\cdot}9 \end{array}$$

The distance from the light-house to the fort is 791·9 yards.

The importance, where any considerable degree of accuracy is required, of the method of solution by calculation, instead of that by construction, will appear from the fact that with tolerably accurate instruments, and some care in the construction, we made the required side, which we here find to be 791·9 yards, to be 800 yards upon the scale; thus committing an error in the construction of about 8 yards.

EXERCISE.

Given in an oblique angled triangle,

$A = 75° \ 30' \ 18''{\cdot}5$, $B = 45° \ 16'$, $c = 1145{\cdot}3$, to find the other parts of the triangle.

Ans. $C = 59° \ 13' \ 41''{\cdot}5$, $a = 1290{\cdot}55$, $b = 946{\cdot}949$.

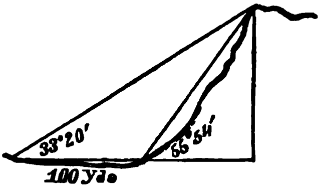
We add another practical

EXAMPLE

Involving the same case of solution combined with the solution of a right angled triangle.

An observer upon a plain desires to find the height of a neighboring hill above the level of the plain.

Near the foot of the hill let him take the angle of elevation to the top, and suppose it to be 55° 54′; then let him measure back a distance, say



100 yards, and again take the angle of elevation, which let be 33° 20′. Then in the triangle of which 100 yards is the base, and 33° 20′ is one of the angles at the base, we may have the angle at the vertex, and opposite to the given base, by observing that the exterior angle of a triangle being equal to the two interior and opposite (Geom. Theorem 13), one of the interior is equal to the exterior, minus the other interior, and therefore the angle at the vertex here is equal to 55° 54′—33° 20′ = 22° 34′; then say, as in the last example,

$$\sin 22^\circ\ 34' : \sin 33^\circ\ 20' :: 100 : \text{side opp. to } 33^\circ\ 20';$$

but in the right angled triangle of which 55° 54′ is the angle at the base, and the height of the hill one of the perpendicular sides, we have the proportion

$$10^{10} : \sin 55^\circ\ 54' :: \text{hypoth.} : \text{height required.}$$

from which, multiplying means and dividing by the first term,

$$\text{height req.} = \frac{\sin 55^\circ\ 54' \times \text{hypoth}}{10^{10}}$$

but from the preceding proportion

$$\text{hypoth. or side op. } 33^\circ\ 20' = \frac{\sin 33^\circ\ 20' \times 100}{\sin 22^\circ\ 34'}$$

substituting this value in the last equation, we have

$$\text{height required} = \frac{\sin 55^\circ\ 54' \times \sin 33^\circ\ 20' \times 100}{10^{10} \times \sin 22^\circ\ 34'}$$

$$\begin{aligned}
\text{arith. comp log. } 10^{10} &= 0{\cdot}00000\\
\text{arith. comp. log. } \sin 22^\circ\ 34' &= 0{\cdot}41594\\
\text{log. } \sin 55^\circ\ 54' &= 9{\cdot}91806\\
\text{log. } \sin 33^\circ\ 20' &= 9{\cdot}73997\\
\text{log. } 100 &= 2\\
\hline
\text{sum rejecting twice } 10 &= 2{\cdot}07397 = \text{log. of } 118\ 6.
\end{aligned}$$

118·6 yards, or 355·8 feet is the height of the hill.

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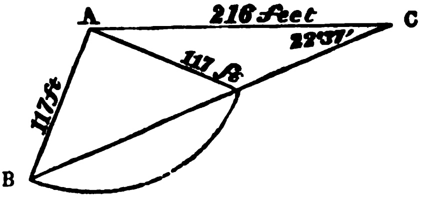
* To this result the height of the eye or of the instrument should be added.

EXAMPLE III.



68. Let there be a street in which the front of a triangular block is 216 feet, and another street making an angle of 22° 37′ with the first; under what angle must a third street be laid out from the extremity A of the first, so that the front of a complete row of buildings upon it shall be 117 feet in length?

SOLUTION.

117 : sin 22° 37′ : : 216 : sin of the angle B

117 arith comp. of log. = 7.93181
22° 37′ log. sin = 9·58497
216 log. = 2·33445
45° 13′ 55″ log. sin = 9.85123

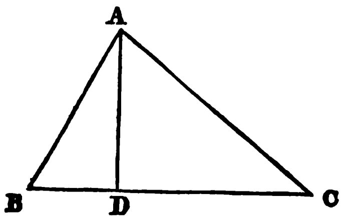
∴ B = 45° 13′ 55″ and 180° — B — C = 112° 9′ 5′ = angle A required.

It must be observed that 9·85123 is also the log. sine of the supplement of 45° 13′ 55″, because the sine of an arc is equal to the sine of its supplement. (Art. 15.) Hence 134° 46′ 5″ is also a value of the angle B, and there are two solutions to the problem, which are both exhibited in the diagram. This case corresponds to Problem 8, Geom.

If the given angle were right or obtuse, there could be but one solution, and the required angle must be acute.

The same is the case if the given side opposite the given angle be greater than the other given side, because in every plane triangle the greater angle is opposite the greater side.

69. We shall next derive a formula for the solution of a triangle when the three sides are given, and one or all of the angles required.



Let ABC be any triangle; from the vertex of one of the angles A, let fall a perpendicular A D upon the side opposite. This perpendicular may fall either within or without the triangle. First suppose that it falls within; then (Geom. Th. 29.)

$$AC^2 = AB^2 + BC^2 - 2\,BC \times BD$$

$$\text{whence } BD = \frac{AB^2 + BC^2 - AC^2}{2\,BC}$$

Also in the right angled triangle A B D, we have

$$R : \cos B :: AB : BD \quad \text{(Art. 38)}$$

whence multiplying the extremes and dividing by the third term

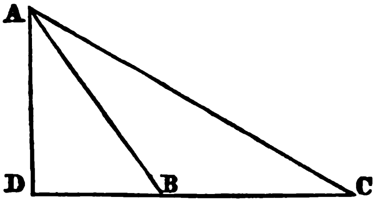
$$\cos B = \frac{R \times BD}{AB}$$

substituting in this expression for B D its value obtained above, we have

$$\cos B = R \times \frac{AB^2 + BC^2 - AC^2}{2\,AB \times BC}$$

an expression for the cosine of an angle in terms of the three sides of a triangle.

Suppose now that the perpendicular falls without the triangle.



Then (Geom. Theorem 28)

$$AC^2 = AB^2 + BC^2 + 2\,BD \times BC$$

hence

$$-BD = \frac{AB^2 + BC^2 - AC^2}{2\,BC}$$

Again, in the right angled triangle A B D

$$R : \cos ABD :: AB : BD$$

hence,

$$\cos ABD = \frac{R \times BD}{AB}$$

But A B D is the supplement of the angle B of the triangle A B C, hence

$$\cos ABD = -\cos B$$

substituting — cos B for cos A B D above, and changing the signs we have

$$\cos B = \frac{-R \times BD}{AB}$$

substituting for $-$ BD in the second member of this equation its value found above, we have as before

$$\cos B = R \times \frac{AB^2 + BC^2 - AC^2}{2\,AB \times BC}.$$

or employing the small letters to represent the sides opposite the angles which are expressed by the large letters of the same name

$$\cos B = R\,\frac{a^2 + c^2 - b^2}{2\,a\,c}$$

That is to say, *the cosine of either angle of a triangle is equal to the sum of the squares of the two sides which contain it, minus the square of the side opposite, divided by twice the rectangle of the containing sides.*

Let us apply this formula to an

EXAMPLE.

Suppose the three sides of a triangular plat of ground are to be 50, 60, and 70 yards, under what angle must the first two be laid out?

$$\text{cos required angle} = R \times \frac{2500 + 3600 - 4900}{2 \times 50 \times 60} = \tfrac{1}{5}$$

if we make $R = 1$; hence $\frac{1}{5}$ or ·20000 is the nat. cosine of the angle required. This angle will be found from the table of sines and cosines, to be 78° 27′ 47″.*

This case might also be solved with the table of log. sines, &c., by subtracting the log. of the whole denominator from that of the whole numerator, and adding 10 the log. of R, or adding at once the arith. comp. of the denominator, the logs. of the numerator and R, rejecting 10 from the sum; in either case the result would be log. cosine of the angle required. The solution is left as an exercise for the student.

70. We now proceed to demonstrate some formulas which express relations between the different trigonometrical lines of the same arc, and

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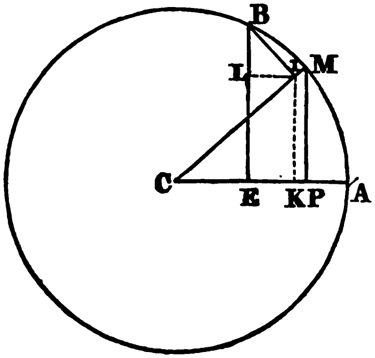
* The seconds are found as follows: Take the difference between the two cosines next greater and next less than yours, and also the difference between yours and the next greater, multiply the latter difference by 60 and divide the product by the former difference; the quotient will be the seconds sought. The reason appears from the following proportion.

diff. of cosines. in the tab.	:	diff. of numbers. 60″	: :	diff. of cosines. yours and the tab.	:	diff. of numbers. seconds required.

between the trigonometrical lines of two different arcs. They are introduced here because necessary for the solution of the few cases of plane triangles which remain. We shall first derive formulas by means of which, when the sines and cosines of two arcs are known, the sine and cosine of their sum or difference may be found. Thus if the sine and cosine of 30°, and also those of 20° be given, those of $50° = 30° + 20°$ or of $10° = 30° - 20°$ may be found.



Let $AM = a$ in the diagram be one of the given arcs, and $BM = b$, be the other. Then $MP = \sin a$, $BI = \sin b$, since it is the perpendicular let fall from one extremity of the arc b, upon the radius which passes through the other extremity. $CP = \cos a$ and CI the distance from the foot of the sine to the centre $= \cos b$. $AB = AM + MB = a + b$, $BE = \sin (a + b)$ and $EC = \cos (a + b)$.

In the triangle CKI we have (Art. 64, IV.)

$$IK = \sin C \times CI$$

or,

$$EL = \sin a \cos b \qquad (1)$$

and in the triangle BIL (Art. 64, V.)

$$BL = BI \cos B$$

or since $B = C$ their sides being respectively perpendicular

$$BL = \sin b \cos a \qquad (2)$$

Adding equations (1) and (2) and observing that $EL + BL = BE = \sin AB$ we have

$$\sin (a + b) = \sin a \cos b + \sin b \cos a \qquad (3)$$

The second member must be understood to be divided by $R = 1$, for a line cannot be equal to the sum of two surfaces (See Art. 64, IX.).

Formula (3) is read thus: *the sine of the sum of any two arcs is equal to the sine of the first into the cosine of the second plus the sine of the second into the cosine of the first, divided by radius.*

Again,

$$CK = \cos . C \times CI = \cos a \cos b \qquad (4)$$

$$EK = IL = \sin B \times BI = \sin a \sin b \qquad (5)$$

Subtracting (5) from (4), and observing that O K — E K = O E = cos A B (Art. 24), we have

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \qquad (6)$$

The second member of (6) must be understood to be divided by R or the first member to be multiplied by R to produce homogeneity.

Formula (6) is read thus: *the cosine of the sum of any two arcs is equal to the rectangle of their cosines minus the rectangle of their sines divided by radius.*

In formula (3) let $a = 60°$ and $b = 20°$ then by the first

$$\sin 80° = \frac{\sin 60° \cos 20° + \sin 20° \cos 60°}{1 \text{ or } 10^{10} \text{ as the case may be}}$$

Performing the operations by the aid of logarithms

$$\begin{array}{llll}
 & \text{log. sin } 60° = & 9.93753 & \\
 & \text{log. cos } 20° = & 9.97299 & \\
\text{log. sin } 20° = 9.53405 & & \overline{} & \\
\text{log. cos } 60° \quad 9{\cdot}69897 & & 9{\cdot}91052^{*} & \text{log. of } {\cdot}81380 \\
\hline
\phantom{\text{log. cos } 60° \quad} 9{\cdot}25302 & & & \text{log. of } {\cdot}17101 \\
 & & & \overline{\sin 80° = {\cdot}98481}
\end{array}$$

By formula (6)

$$\cos 80° = \frac{\cos 60° \cos 20° - \sin 60° \sin 20}{1 \text{ or } 10^{10} \text{ or whatever R may be}}\dagger$$

We shall derive expressions for $\sin(a-b)$ and $\cos(a-b)$ or the sine and cosine of the difference of two arcs in terms of the arcs themselves, by making, in the formulas just derived, for $\sin(a+b)$ and $\cos(a+b)$; $b = -b$, observing that $\cos(-b) = \cos b$ and $\sin(-b) = -\sin b$ (Art. 27). By this substitution there results

---

[*] From each of the logs. 9·91052 and 9·25302, 10 must be rejected in order to pass from the table of logarithmic sines, &c., in which the radius is 10000000000, the logarithm of which is 10, to the table of natural sines, &c., in which the radius is 1. The characteristics of these logarithms would thus become $\bar{1}$, but they may be read so as they stand according to Art. 54, Ex. 2.

[†] The student may develope this as an exercise, and compare the result with cos 80° as given by the table of natural sines and cosines.

$$\sin (a - b) = \sin a \cos b - \sin b \cos a \qquad (7)$$

and

$$\cos (a - b) = \cos a \cos b + \sin a \sin b \qquad (8)$$

EXERCISES.

Find the sine of 57° from sin of 15° = ·25882, and cos = ·96593, and sin 42° = ·66913, and cosine = ·74314.

Find the cosine of 9° from sine and cosine of 24° = ·40674 and ·91355 and sine and cosine 15° = ·25882 and ·96593.

These four formulas for the sine and cosine of the sum and difference of two arcs should be committed to memory, as they are constantly recurring in trigonometry, and in the higher analysis. The four may be expressed in two by the use of the double sign, thus

$$\sin (a \pm b) = \sin a \cos b \pm \sin b \cos a$$

$$\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$$

71. From formula (3) sin $(a+b)$ = &c., we derive one much used in the higher analysis for expressing twice an arc in terms of the arc itself, by simply making $b = a$ the result is

$$\sin 2a = 2 \sin a \cos a \qquad (1)$$

the two terms of the second member becoming the same.

We also get an analogous expression for the cosine of twice an arc by making $b = a$ in formula (6) of the last article cos $(a + b)$ = &c. This expression is

$$\cos 2 a = \cos^2 a - \sin^2 a \qquad (2)$$

Thus knowing the sine and cosine of 20°, these last two formulas would give us the sine and cosine of 40°.

These two formulas may be modified so as to express the sine and cosine of an arc in terms of half the arc, under which last form they are much used. This is accomplished by making $a = \frac{1}{2}\ a$, which is legitimate, since a is supposed to have no particular value; then $2\ a$ becomes a and we have from (1)

$$\sin a = 2 \sin \tfrac{1}{2}\, a \cos \tfrac{1}{2}\, a \qquad (3)$$

and from (2)

$$\cos a = \cos^2 \tfrac{1}{2}\, a - \sin^2 \tfrac{1}{2}\, a \qquad (4)$$

EXERCISES.

Find the sine of 32° from the sine and cosine of 16°, the former being ·27564 and the latter ·96126. Find also the cosine of 32° from the same data.

NOTE.—This last is best done by observing that the product of the sum and difference is equal to the difference of the squares.

72. By means of this last, and a very simple formula depending upon the well-known property of the right angled triangle, that the square of the hypothenuse is equal to the sum of the squares of the other two sides, a formula expressing the value of the sine of half an arc in terms of the arc itself may be obtained.

The formula depending upon the property of the right angled triangle, will be found by referring to the last diagram, in which the triangle O P M is right angled at P, whence (Geom. Th. 26).

$$\text{O P}^2 + \text{P M}^2 = \text{O M}^2$$

or calling A M $\frac{1}{2} a$

$$\cos^2 \tfrac{1}{2} a + \sin^2 \tfrac{1}{2} a = \text{R}^2 \qquad (5)$$

Introducing R into equation (4) according to the rule for homogeneity at Art. 31, and changing the order of the members it becomes

$$\cos^2 \tfrac{1}{2} a - \sin^2 \tfrac{1}{2} a = \text{R} \cos a \qquad (6)$$

Subtracting (6) from (5) we have

$$2 \sin^2 \tfrac{1}{2} a = \text{R}^2 - \text{R} \cos a$$

Dividing by 2 and taking the square root of both members of this last equation, we have the formula required.

$$\sin \tfrac{1}{2} a = \sqrt{\tfrac{1}{2} \text{R}^2 - \tfrac{1}{2} \text{R} \cos a} \qquad (7)$$

making R = 1

$$\sin \tfrac{1}{2} a = \sqrt{\tfrac{1}{2} - \tfrac{1}{2} \cos a} \qquad (8)$$

73. We resume the solution of triangles, having now a formula, by means of which we shall be able to derive an expression for one of the angles of a triangle in terms of the three sides; an expression which will be found much more convenient for the application of logarithms than that contained in Art. 69.

By Art. 69, making R = 1

$$\cos \text{B} = \frac{a^2 + c^2 - b^2}{2\,a\,c} \qquad (1)$$

putting B in the place of a in the formula for sin $\frac{1}{2}$ a (formula (8) of the last article) we have

$$\sin \tfrac{1}{2} B = \sqrt{\tfrac{1}{2} - \tfrac{1}{2} \cos B}$$

substituting for cos B in this, its value in (1), we have

$$\sin \tfrac{1}{2} B = \sqrt{\tfrac{1}{2} - \tfrac{1}{2} \frac{a^2 + c^2 - b^2}{2\,a\,c}}$$

reducing the terms under the radical to a common denominator, there results

$$\sin \tfrac{1}{2} B = \sqrt{\frac{2\,a\,c - a^2 - c^2 + b^2}{4\,a\,c}}$$

but (Alg. Art. 13, Note 2)

$$2\,a\,c - a^2 - c^2 = -(a - c)^2$$

hence

$$\sin \tfrac{1}{2} B = \sqrt{\frac{b^2 - (a - c)^2}{4\,a\,c}}$$

but the difference of the squares of two quantities is equal to the product of their sum and difference (Alg. Art. 13, Note 2), hence

$$b^2 - (a - c)^2 = (b + a - c)\,(b - a + c)$$

substituting the second member of this in place of the first in the preceding equation, and separating the 4 of the denominator into two factors 2 × 2, we have

$$\sin \tfrac{1}{2} B = \sqrt{\left\{ \frac{\frac{(b + a - c)}{2} \; \frac{(b + c - a)}{2}}{a\,c} \right\}}$$

but

$$\frac{b + a - c}{2} = \frac{b + a + c}{2} - c \text{ and } \frac{b + c - a}{2} = \frac{b + c + a}{2} - a$$

representing $b + a + c$ the sum of the three sides of the triangle by s, the second members of the two last equalities become

$$\tfrac{1}{2}\,s - c \text{ and } \tfrac{1}{2}\,s - a$$

substituting these for their equals in the preceding equation it becomes

$$\sin \tfrac{1}{2}\,\text{B} = \sqrt{\frac{(\frac{1}{2}s - c)\,(\frac{1}{2}s - a)}{a\,c}} \qquad (2)$$

the formula sought.*

As the angles have each the same relations to the corresponding sides of a triangle, the same formula by a proper modification will furnish the values of the angles A and C.

It may be expressed in ordinary language thus, *the sine of half either angle of a triangle is equal to radius into the square root of half the sum of the three sides minus one of the adjacent sides, into half the sum minus the other adjacent side, divided by the rectangle of the adjacent sides.*

To apply this to an

EXAMPLE.

Let there be three places at distances from each other respectively of 50, 60, and 70 miles. Required the angle under which two roads must depart from that which is 60 and 70 distant from the other two, in the direction of these last. 60 and 70 will be the sides of a triangle adjacent the required angle, and 50 the side opposite; then

$$\sin \tfrac{1}{2} \text{ the angle} = \text{R}\sqrt{\frac{(\frac{1}{2}s - 70)\;(\frac{1}{2}s - 60)}{60 \times 70}}$$

$$\tfrac{1}{2}s = \frac{180}{2} = 90,\ \tfrac{1}{2}s - 70 = 20$$

$$\text{and } \tfrac{1}{2}s - 60 = 30$$

$$\begin{aligned} \text{log. of } 20 &= 1{\cdot}30103 \\ \text{log. of } 30 &= 1{\cdot}47712 \\ \text{ar. comp. of log. of } 60 &= 8{\cdot}22185 \\ \text{ar. comp. of log. of } 70 &= 8{\cdot}15490 \\ \hline \text{sum rejecting twice } 10 &= \bar{1}{\cdot}15490 \end{aligned}$$

Divide this sum by 2 for √, quot. = — 0·5 + ·07745

Add 10 to multiply by R sum = 9·57745 = log. sin of ½ the required angle.

---

* Radius must be understood as a factor of the second member for the sake of homogeneity, since the quantity under the radical is the ratio of a surface to a surface, and therefore an abstract number.

From the tables we find ½ the angle to be 22° 12′ 28″.

The whole angle required will be double this or

44° 24′ 56″

The other two angles may be found in a similar manner; the one is 67° 07′ 18″, the other 78° 27′ 40″.

If R in the above formula should be made to pass (by squaring it) under the radical sign, it would be necessary to add twice 10 in order to effect the multiplication by this factor R^2 before taking the square root. But as on the other hand twice 10 must be rejected for the arithmetical complements used, these two operations exactly counterbalance each other, and neither of them need be performed. By adding together the four logarithms, therefore, and dividing by 2, the same result will be obtained. The operation in the above example would be as follows:

log. of 20 = 1·30103
log. of 30 = 1·47712
ar. comp. of log. of 60 = 8·22185
ar. comp. of log. of 70 = 8·15490
2)19·15490
log. sin. of 22° 12′ 28″ = 9·57745

The best mode of proceeding in the solution of a plane triangle when three sides are given, is to prepare a blank form similar to that on p. 58, by ruling four columns, the first for the arguments, and each of the other three for the trigonometrical functions of those arguments necessary to be employed in the calculation of one of the three angles. Thus, a, b, c, denoting the given sides, s their sum, and A, B, C the required angles,

Column of Arguments.	Column for calculation of A.	Column for calculation of B.	Column for calculation of C.
a		ar. co. log.	ar. co log.
b	ar. co. log.		ar. co. log.
c	ar. co. log.	ar. co. log.	
s			
$\frac{1}{2}s$			
$\frac{1}{2}s - a$		log.	log.
$\frac{1}{2}s - b$	log.		log.
$\frac{1}{2}s - c$	log.	log.	
	2)		
½ A	log. sin	2)	
½ B		log. sin	2)
½ C			log. sin

To show how this form is filled up take the following example.

Given $a = 33$, $b = 42{\cdot}6$, $c = 53{\cdot}6$, required A, B, C.

Column of Arguments.	Column for calculation of A.	Column for calculation of B.	Column for calculation of C.
a 33		ar. co. log. 8·4814861	ar. co. log. 8·4814861
b 42·6	ar. co. log. 8·3705904		ar. co. log. 8·3705904
c 53·6	ar. co. log. 8·2708352	ar. co. log. 8·2708352	
s 129·2			
$\frac{1}{2}s$ 64·6			
$\frac{1}{2}s-a$ 31·6		log. 1·4996871	log. 1·4996871
$\frac{1}{2}s-b$ 22	log. 1·3424227		log. 1·3424227
$\frac{1}{2}s-c$ 11	log. 1·0413927	log. 1·0413927	
	2)19·0252410		
½ A 18° 59′ 56″·5	log. sin 9·5126205	2)19·2934011	
½ B 26° 18′ 53″·3		log. sin. 9·6467005	2)19·6941863
½ C 44° 41′ 10″			log. sin 9·8470931

A = 37° 59′ 53″, B = 52° 37′ 46″·6, C = 89° 22′ 20″.

There are other forms which have some advantages over the above, and which may be derived in an analogous manner. They are as follows.

$$\sin A = \frac{2R}{bc}\sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s-a)\,(\tfrac{1}{2}s-b)\,(\tfrac{1}{2}s-c)} \qquad (3)$$

$$\cos \tfrac{1}{2}A = R\sqrt{\frac{\frac{1}{2}s\,(\frac{1}{2}s-a)}{bc}}{}^{*} \qquad (4)$$

$$\tan \tfrac{1}{2}A = R\sqrt{\frac{(\frac{1}{2}s-b)\,(\frac{1}{2}s-c)}{\frac{1}{2}s\,(\frac{1}{2}s-a)}}{}^{\dagger} \qquad (5)$$

The student may write the blank forms for these formulas as an exercise.

EXAMPLES.

1. Given in a plane triangle the three sides 120, 112·65, and 112, to find the three angles.

Ans. 57° 27′, 57° 58′ 39″, 64° 34′ 21″

---

[*] Derived from (5) and (6) of Art. 72, and (1) of Art 73. A convenient form.

[†] By dividing the formula $\sin \frac{1}{2}A = \sqrt{\frac{1}{2} - \frac{1}{2}\cos A}$ by the formula $\cos \frac{1}{2}A = \sqrt{\frac{1}{2} + \frac{1}{2}\cos A}$ is found $\tan \frac{1}{2}A = \frac{\sin A}{1+\cos A}$ from which (4) above may be derived; or at once dividing (3) by (4). In a similar manner may be found $\cot \frac{1}{2}a = \frac{\sin a}{1-\cos a}$, $\sec \frac{1}{2}a = \sqrt{\frac{2\sec a}{\sec a+1}}$, and $\operatorname{cosec} \frac{1}{2}a = \sqrt{\frac{2\sec a}{\sec a-1}}$.

2. Given $a = 6876$, $b = 4231$, $c = 8913{\cdot}24$, to find A, B, and C.

$$\textit{Ans.}\left\{\begin{array}{l} \text{A} = 48^\circ\ 24'\ 42'' \\ \text{B} = 27\ \ 24\ \ 6 \\ \text{C} = 104\ \ 11\ \ 18 \end{array}\right.$$

74. Before treating of the only remaining case in the solution of triangles, it will be convenient to demonstrate some additional general formulas which shall present certain important relations of the trigonometrical lines of two different arcs; which formulas are of frequent use in the higher analysis, are employed in the subsequent parts of the work, and will be immediately of service in deriving a formula for the last case of plane trigonometry which we have to consider.

Add together equations (3) and (7) of (Art. 70) which express the values of $\sin(a+b)$ and $\sin(a-b)$ and the resulting equation, cancelling the second terms of the second members which are similar with contrary signs, is

$$\sin(a+b) + \sin(a-b) = 2\sin a \cos b \qquad (1)$$

make

$$a + b = p \text{ and } a - b = q$$

add these last two equations; there results

$$2a = p + q \text{ whence } a = \tfrac{1}{2}(p+q)$$

subtracting the same equation, the second from the first,

$$2b = p - q \text{ whence } b = \tfrac{1}{2}(p-q)$$

substituting in equation (1) the values of $a+b$, $a-b$, a and b in terms of p and q, that equation becomes

$$\sin p + \sin q = 2\sin\tfrac{1}{2}(p+q)\cos\tfrac{1}{2}(p-q)\ ^{*} \qquad (2)$$

Which may be translated into ordinary language thus: *the sum of the sines of two arcs is equal to twice the sine of half the sum into the cosine of half the difference of those arcs.*

By subtracting the latter of the same equations (3) and (7) of (Art. 69) from the former, and reducing similar terms, there results

$$\sin(a+b) - \sin(a-b) = 2\sin b\cos a \qquad (3)$$

---

* R must be understood either as a divisor of the second member or multiplier of the first, because the sum of two lines cannot be equal to a rectangle.

making the same substitutions as above in equation (1) this last equation becomes

$$\sin p - \sin q = 2 \sin \tfrac{1}{2}(p-q) \cos \tfrac{1}{2}(p+q) \qquad (4)$$

or, *the difference of the sines of two arcs is equal to twice the sine of half their difference into the cosine of half their sum.**

Divide equation (2) by equation (4)

$$\frac{\sin p + \sin q}{\sin p - \sin q} = \frac{\sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p+q) \sin \frac{1}{2}(p-q)} \qquad (5)$$

but

$$\frac{\sin \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p+q)} = \text{tang } \tfrac{1}{2}(p+q) \text{ (Art. 32) and}$$

$$\frac{\sin \frac{1}{2}(p-q)}{\cos \frac{1}{2}(p-q)} = \text{tang } \tfrac{1}{2}(p-q)$$

or inverting this last

$$\frac{\cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p-q)} = \frac{1}{\tan \frac{1}{2}(p-q)}$$

substituting in (5) the values of $\frac{\sin \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p+q)}$ and $\frac{\cos \frac{1}{2}(p-q)}{\sin \frac{1}{2}(p-q)}$

the equation becomes

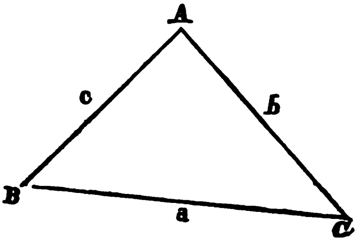
$$\frac{\sin p + \sin q}{\sin p - \sin q} = \frac{\tan \frac{1}{2}(p+q)}{\tan \frac{1}{2}(p-q)} \qquad (6)$$

which may be expressed in a proportion thus: *the sum of the sines of any two arcs is to the difference of their sines as the tangent of half their sum is to the tangent of half their difference.*

75. Let A B C be any triangle; then (Art. 67)

$$a : b :: \sin \text{A} : \sin \text{B}$$

or by composition, (Alg. Art. 133, IX., Geom. Theorem 47),



$$a + b : a - b :: \sin \text{A} + \sin \text{B} : \sin \text{A} - \sin \text{B}$$

---

* The same remark applies to this form as to (2).

but by equation (6) Art. 74.

$$\sin A + \sin B : \sin A - \sin B :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B)$$

hence,

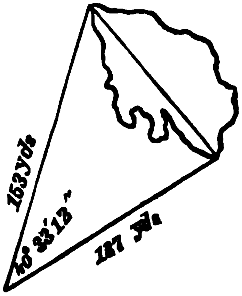
$$a + b : a - b :: \tan \tfrac{1}{2}(A + B) : \tan \tfrac{1}{2}(A - B)$$

That is to say, *the sum of two of the sides of a plane triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.*

76 This proportion is employed when two sides and the included angle of a triangle are given to find the other parts. Since the three angles of every triangle are together equal to two right angles or 180°, subtracting the given included angle from 180°, the remainder is the sum of the two angles opposite the given sides; then substituting for a and b in the above proportion the two given sides, three terms of it are known and the fourth may be found. After which, having half the sum and half the difference of the unknown angles, these angles themselves can be found by adding half the sum to half the difference for the greater, and subtracting half the difference from half the sum from the lesser; when all the parts of the triangle will be known, except one side, which may be found by the proportion *the sines of the angles are as the opposite sides.* (Art. 67.)

EXAMPLE.

An observer wishing to know the length of a small lake, measured two lines from the same point to the two extremities of the lake, which he found to be respectively 153 and 137 yards; he also observed with an instrument for taking angles the angle subtended from this point by the lake to be 40° 33′ 12″.



SOLUTION.

I.

To find the two other angles.

As sum of the given sides 290 = arith. comp. of log.$^{-1}$[*] 7·53760

: diff. of the sides 16 = log.$^{-1}$ 1·20412

: : tan of ½ sum op. angs. = ½ (180 — 40° 33′ 12″) = 69° 43′ 24″ = log.$^{-1}$ 10·43245

: tan of ½ diff. of op. angs. log. of which is sum rejec. 10 = 9·17417 = log. tan 8° 29′ 37″

Add and subt. with ½ sum of angs. 69° 43′ 24″

sum = greater angle 78° 13′ 1″

diff. = lesser angle 61° 13′ 47″

II.

To find the remaining side.[†]

As sin 78° 13′ 1″ = arith comp. log.$^{-1}$ 0·00925

: opp. side 153[‡] = log.$^{-1}$ 2·18469

: : sin 40° 33′ 12″ = log.$^{-1}$ 9·81302

: side opp. 40° 30′ 12″ =

log.$^{-1}$ sum rejec. 10 = 2·00696 log of 101·616.

The length of the lake is 101·616 yards.

The blank form for this case would be as follows:

a		log.
b		
$a + b$	ar. co log.	
$a - b$	log.	
C		log. sin
$A \div B$		
$\frac{1}{2}(A + B)$	log. tan	
$\frac{1}{2}(A - B)$	log. tan	
A		ar. co. log. sin
B		
c		log.

---

[*] This notation "log.$^{-1}$" signifies *the number whose log. is.*

[†] A more direct mode of finding this side when it is the only part required, is given at Art. 77.

[‡] 153 is known to be the side opposite 78° 13′ 1″ because the greater angle of a triangle is always opposite the greater side. (Geom. Theorem 9.)

The same form filled up with the given example above is given below.

a	153		log. 2·18469
b	137		
$a+b$	290	ar. co. log. 7·53760	
$a-b$	16	log. 1·20412	
C	40° 33′ 12″		log. sin 9·81302
A + B	139° 26′ 48″		
½ (A + B)	69° 43′ 24″	log. tan 10·43245*	
½ (A — B)	8° 29′ 37″	log. tan 9·17417	
A	78° 13′ 1″		ar. co. log. sin 0·00925
B	16° 13′ 47″		
c	101·616		log. 2·00696

77. Given two sides and the included angle of a plane triangle to determine the third side, without finding the remaining angles.

The general expression for the side c, in terms of the two sides a, b, and the included angle C, is (Art. 69) on the supposition of R = 1,

$$\begin{aligned} c^2 &= a^2 + b^2 - 2\,ab \cos \mathrm{C} \\ &= (a-b)^2 + 2\,ab\,(1 - \cos \mathrm{C}) \\ &= (a-b)^2 + 2\,ab \cdot 2 \sin^2 \tfrac{1}{2}\,\mathrm{C} \\ &= (a-b)^2 \left\{ 1 + \frac{4\,ab}{(a-b)^2} \sin^2 \tfrac{1}{2}\,\mathrm{C} \right\} \end{aligned} \qquad (1)$$

Assume the second term within the brackets equal to $\tan^2\theta$, then, since $1 + \tan^2\theta = \sec^2\theta = \dfrac{1}{\cos^2\theta}$, we have

$$c = (a-b)\,\frac{\text{rad}}{\cos\theta} \qquad (2)$$

Hence c is determined by these two formulas, viz.,

$$\log.\tan\theta = \log.2 + \tfrac{1}{2}\log.a + \tfrac{1}{2}\log.b + \log.\sin\tfrac{1}{2}\,\mathrm{C} - \log(a-b)$$
$$\log.c = \log.(a-b) + 10 - \log.\cos\theta.\dagger$$

---

* Log. cot ½ C might be used instead of log. tan ½ (A + B) since 180° — C = A + B and ∴ 90° — ½ C = ½ (A + B) and tan (90° — ½ C) = cot ½ C. But nothing would be gained, since ½ (A + B) must be employed to add and subtract with ½ (A — B)

† When b is nearly equal to a, the following formulas will give c with greater exactness. For demonstration see App. I. Art. 23.

$$\sin\phi = \frac{2\sqrt{ab}}{a+b}\cos\tfrac{1}{2}\,\mathrm{C} \qquad (3)$$

$$c = (a+b)\cos\phi \qquad (4)$$

EXAMPLE.

Given *a* = 562, *b* = 320, and c = 128° 4′, to find *c*.

log. 2	0·30103		
½ log. 562	1·37487		
½ log. 320	1·25257		
log. sin 64° 2′	9·95378		
ar. comp. log. 242	7·61618	log. 242 + 10	12·38382
log. tan θ	10·49843	∴ log. cos θ =	9·48072
log. *c*	800·01		2·90310

EXERCISES.

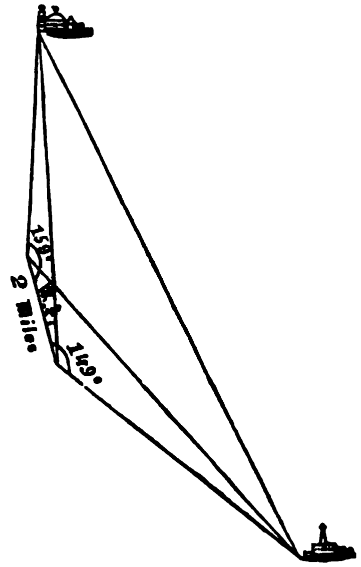
Given *a* 5891, *b* 4562·34, c 30° 20′ 10″·3 to find the other parts of the triangle.
Ans. A 99° 57′ 5″·25, B 49° 42′ 44·45, *c* 3020·823.

Given A 40° 55′ 31″, *b* 83·25, *c* 100, to find *a*.
Ans. *a* 65·9574.

This case of the solution of a triangle combined with that exhibited at Art. 67, serves to determine the horizontal distance between two inaccessible objects.

Let the distance between two towns which are in sight be required.

Measure a line upon the ground (which is called the base line) of 2 miles. Take the angles at each extremity formed by this base line and a line to each of the towns. Two triangles will be formed, in each of which a side, viz. the base line, and two adjacent angles will be given. Let the angles in the triangle of which the upper town is the vertex be 159° and 14°; and those in that of which the lower town is the vertex be 25° and 149°. Calculate the distance from one extremity of the base line, say the upper extremity, to each of the towns, as in Art. 67. Then you will know two sides of a triangle the third side of which is the distance between the towns required.



The included angle between these two sides is 159° — 25° = 134°.

Having then two sides and the included angle, the remainder of the solution is the same as in the last case.

We leave it as an exercise for the learner. *Ans.* 13·08 miles.

MISCELLANEOUS EXAMPLES.

(1.) From the equation $\sin^2 a + 5 \cos^2 a = 3$ to find the value of $\sin a$.

Ans. $\sin a = \frac{1}{\sqrt{2}}$

(2.) If $\sin^2 a = m \cos a - n$ to determine $\cos a$.

Ans. $\cos a = -\frac{1}{2} m \pm \sqrt{\frac{1}{4}m^2 + n + 1}$.

(3.) Given $\sin a = m \sin b$, $\tan a = n \tan b$ to find $\sin a$ and $\cos b$.

Ans. $\sin a = \sqrt{\frac{m^2 - n^2}{1 - n^2}}$ $\cos b = \frac{n}{m}\sqrt{\frac{1 - m^2}{1 - n^2}}$

(4.) Prove $\sin (a + b + c) = \sin a \cos b \cos c + \sin b \cos a \cos c + \sin c \cos a \cos b - \sin a \sin b \sin c$.

(5.) Given $\tan^2 a + 4 \sin^2 a = 6$ to find a.

Ans. $a = 60°$.

(6.) Sin $a = \sin 2a$ to find $\sin a$.

Ans. $\sin a = \sqrt{\frac{3}{4}}$

(7.) Given the base of a triangle 87·75, the vertical angle 73° 20′, and the difference of the angles at the base 13° 4′, to find the other parts.

Ans. { 46° 48′, 59° 52′.
79·2194, 66·7721.

(8.) Given the base 117·3, the vertical angle 19° 18′, and the ratio of the other two sides 8 : 11 to solve the triangle.

Aus. 123° 13′ 23·″7, 37° 28′ 36″·3, 215·94, 296·89.

(9.) Find into what two parts the perpendicular from the vertex of the angle C, divides the side c of a triangle when A = 33°, C = 75°, and a = 2134.

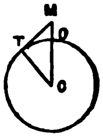
Ans. 3125·2 and 659·44.

(10.) The side of a regular polygon of 41 sides is 0·736. What is the ratio of the radii of the inscribed and circumscribed circles?

Ans. ·99708.

(11.) Find the extent of the circle of vision, viz. the arc TO, from the top of a mountain M, whose height is 5460 feet, supposing the radius of the earth to be 3956·1 miles.

Ans. 91 miles, 1558·88 feet.



(12.) When the sun's altitude is 19° 16′, the peak of a mountain casts its shadow at a certain point, and at another point when the sun's altitude is 20° 42′, distant from the former point 937 ft. Now supposing the sun to be vertical on the top of the mountain at noon, what is its height?

Ans. 4369.1 ft.

(13.) Two forces, one of 410, the other of 320 pounds, act under an angle of 51° 37′, required the direction and intensity of their resultant.*

Ans. The resultant makes an angle of 29° 13′ 46″·7, with the less force, and 22° 23′ 13″·3 with the greater.
Intensity 702·39838 pounds.

(14.) From the edge of a ditch, the width of which was 36 feet, the angle of elevation to the top of an opposite wall was 62° 40′; to find the height of the wall and the length of a ladder which would reach obliquely across the ditch to the top of the wall

Ans. Height of wall, 69·64.
Ladder, 78·4 feet.

(15.) To find the length of a shoar, which, projecting 11 feet from the perpendicular face of a building, will support a jamb 23 feet 10 inches above the ground ?

Ans. 26 feet 3 inches.

(16.) Suppose that a ladder, 40 feet long, will reach a window 33 feet from the ground on one side of a street, and on being turned over, without moving the foot, it will reach a window 21 feet high on the other side ; to find the breadth of the street ?

Ans. 56·649 feet.

(17.) A liberty-pole whose top was broken off strikes the ground at 15 feet distance from the foot of the pole ; to find the height of the whole if the broken piece measures 39 feet in length ?

Ans. 75 feet.

(18.) At 170 feet distance from the bottom of a tower, suppose the angle of elevation to be 52° 30′ ; to find the altitude of the tower ?

Ans. 221 feet.

(19.) From the top of a tower by the sea-shore, 143 feet high, the angle of depression of a ship was observed to be 35° ; to find the distance of the ship from the bottom of the tower ?

Ans. 204·22 feet.

(20.) To find the height of a hill, the angle of elevation at the bottom being 46°, and 200 yards distant from the bottom 31° ?

Ans. 286·28 yards.

(21.) To find the height and distance of an inaccessible tower, on a horizontal plane, the angle of elevation being 58°, and at a point 300 feet more distant, the angle being only 32° ?

Ans. Height, 307·53.
Distance, 192·15.

(22.) To find the height of a tower on the top of an inaccessible hill, the angle of elevation to the top of the hill being 40°, the top of the tower 51°, and 200 feet further back the angle to the top of the tower being 33° 45′ ?

Ans. 93·33148 feet.

(23.) From a window on a level with the bottom of a steeple, the angle of elevation of the top of the steeple being 40° ; and from another window, 18 feet directly above the former, the angle was 37° 30′ ; to find the height and distance of the steeple ?

Ans. Height, 210·44.
Distance, 250·79.

---

* The resultant of two forces is a single force equivalent to them, and is the diagonal of a parallelogram of which the two forces given are sides.

(24.) A balloon being directly over one of two towns whose distance apart was 8 miles, the angle of depression of the second was observed to be 10°. Required the height of the balloon?

Ans. 1·41 of a mile.

(25.) The horizontal angles were observed from each of two stations, 3000 feet apart, by first sighting to the other station, and then to a balloon, and the angle of elevation at one, as follows:

1st Station.	Hor. angle,	75° 25′	2d. 64° 30′
	Angle of elev.	18°	

Required the height and horizontal distance of the balloon from the first station.

Ans. Distance, 4205 feet.
Height, 1366 feet.

(26.) Two vessels of war anxious to cannonade a fort, are so remote from it that their guns cannot reach it with effect. In order to find the distance they move a quarter of a mile apart, then each vessel observes and measures the angles which the other and the fort subtend: the angles being 83° 45′, and 85° 15′, required the distance between each vessel and the fort?

Ans. 2292·26
2298·05 yards.

(27.) Wishing to know the distance to an object on the other side of a river, I measured a base line of 400 feet in a right line by the side of the river, and found that the two angles, one at each end of this line, subtended by the other end and the object, were 68° 2′ and 73° 15′. Required the distance between each station and the object?

Ans. 593·08
612·38 feet.

(28.) Wanting to know the breadth of a river, I measured a base line of 500 feet in a right line close by its bank; the angles subtended by lines connecting each extremity of this line and an object on the opposite bank, were 53° and 79° 12′. Required the perpendicular breadth of the river?

Ans. 529·48 feet.

(29.) Suppose it be required to find the distance between two headlands, measure from each of them to any point inland, and supposing the distances respectively to be 735 feet and 840 feet, also the horizontal angle subtended between these two lines to be 55° 40′, what was the required distance?

Ans. 741·2 feet.

(30.) Wishing to know the distance between a church and tower, situated at a distance on the other side of a river, I measured a base line along the side where I was, of 600 feet, and at each end of it took the angles subtended by the other end and the church and tower; at one end the angles were 58° 20′ and 95° 20′, and at the other end the angles were 53° 30′, and 98° 45′. Required the distance?

Ans. 959·5866 feet.

(31.) To determine the intensity and direction of a force which, combined with another force expressed by 128, shall produce a resultant of 200, which shall make an angle with the direction of the given force of 18° 24′.

Ans. Intensity, 88·32714. Angle, 27° 13′ 16″·6.

(32.) To determine the force with which a body weighing 516 pounds moves down a plane inclined to the horizon under an angle of 14° 10′.[*]

Ans. 126·288 lbs.

(33.) The angle of incidence of a ray of light falling upon a surface being 46°, and the angle of refraction being 35° 11′, to find the index of refraction.

Note.—The index of refraction is the ratio of the sine of the angle of incidence to the sine of the angle of refraction.

Ans. ·8.

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[*] The force down an inclined plane is to the force of gravity, as the height of the plane is to its length.

APPENDIX I.

1. We have postponed to this place the investigation of a few formulas requisite for the study of Analytical Geometry, with other matters of interest.

By resuming the expression for the tangent (Art. 32), and putting $a + b$ for a, we have

$$\tan(a \pm b) = \frac{\sin(a \pm b)}{\cos(a \pm b)}$$

But by Art 70

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

and

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

substituting these values the first equation becomes

$$\tan(a \pm b) = \frac{(\sin a \cos b \pm \sin b \cos a)}{\cos a \cos b \mp \sin a \sin b}$$

dividing both numerator and denominator of the second member of this equation by $\cos a \cos b$ we have

$$\tan(a \pm b) = \frac{\left(\frac{\sin a}{\cos a} \pm \frac{\sin b}{\cos b}\right)}{1 \mp \frac{\sin a \sin b}{\cos a \cos b}}$$

substituting for $\frac{\sin a}{\cos a}$ and $\frac{\sin b}{\cos b}$ their values (Art. 32) $\tan a$ and $\tan b$, the last expression becomes

$$\tan(a \pm b) = \frac{(\tan a \pm \tan b)}{1 \mp \tan a \tan b} \tag{1}$$

i. e., *the tangent of the sum or difference of two arcs is equal to rad. square into the sum or difference of their tangents divided by rad. square minus or plus the rectangle of their tangents.*[*]

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[*] The mode in which R^2 enters is derived from the principle of homogeneity.

If a represent the tangent of a and a' the tangent of a', then

$$\tan(a - a') = \frac{a - a'}{1 + aa'}\ ^{*}$$

Using the upper sines and making $b = a$ in equation (1), we have

$$\tan 2a = \frac{2 \tan a}{1 - \tan^2 a}. \qquad (2)$$

Tan $3a$, tan $4a$, &c., may be found by making b successively equal to $2a$, $3a$, &c.

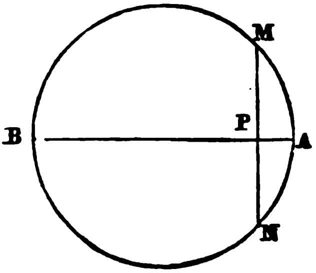
EXERCISE.

Prove $\tan(a + b + c) = \dfrac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan a \tan c - \tan b \tan c}$

2. The sine and cosine of 45° are equal, since the complement of 45° is 45°. These two lines form two sides of a right-angled triangle of which radius is the hypothenuse.

$$\therefore \sin 45^\circ = \cos 45^\circ = \frac{R}{\sqrt{2}} = \tfrac{1}{2} R\sqrt{2}$$

3. *The sine of $\frac{1}{2}$ an arc is equal to $\frac{1}{2}$ the chord of the arc.*



For let MN be the arc ; draw the diameter BA perpendicular to the chord MN of this arc ; this perpendicular bisects the chord, and also the arc subtended by it (Geom. Theorem 34), but MP half the chord is the sine of MA half the arc, since MP is a perpendicular from one extremity M to the diameter which passes through the other extremity A.

Corollary.—The chord of 60°, or $\frac{1}{6}$ of the circumference, which is the side of the regular hexagon, is equal to R (Geom. Prob. 31), hence the sine of 30° is equal to $\frac{1}{2}$ R.

Again,

$$\cos 30^\circ = \sqrt{1 - \sin^2 30^\circ}$$
$$= \sqrt{1 - \tfrac{1}{4}}$$
$$= \frac{\sqrt{3}}{2}$$

Also,

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ}$$
$$= \frac{1}{\sqrt{3}}$$
$$\cot 30^\circ = \frac{1}{\tan 30^\circ}$$
$$= \sqrt{3}$$

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* The form employed in Analytical Geometry.

4. Referring to Art. 33, it will be observed that

$$\sec = \frac{R^2}{\cos}$$

but,

$$\cos 60^\circ = \sin 30^\circ = \tfrac{1}{2}\,R,$$

hence,

$$\sec 60^\circ = \frac{R^2}{\tfrac{1}{2}\,R} = 2\,R = \textit{the diameter of the circle.}$$

To find the numerical value of the sine, cosine, &c. of 60°.

$$\begin{aligned}\sin 60^\circ &= \cos (90^\circ - 60^\circ)\\ &= \cos 30^\circ\\ &= \frac{\sqrt{3}}{2}, \text{ by last art.}\end{aligned}$$

Again,

$$\begin{aligned}\cos 60^\circ &= \sin (90^\circ - 60^\circ)\\ &= \sin 30^\circ\\ &= \frac{1}{2}\end{aligned}$$

Also,

$$\tan 60^\circ = \sqrt{3}$$

$$\cot 60^\circ = \frac{1}{\sqrt{3}}$$

5. Making $a = 45^\circ$ in equation (1) of Art. 1, App. I., we have

$$\tan (45^\circ \pm b) = \frac{1 \pm \tan b}{1 \mp \tan b}.^*$$

6. Our demonstration for the sine and cosine of the sum of two arcs at Art. 70, might seem to want generality, since the arcs a and b are there supposed to be less than 90°. That these arcs may extend to the other quadrants, can be shown as follows:

Let $a = 90^\circ + m$, then will the formula

$$\sin (a + b) = \sin a \cos b + \sin b \cos a \qquad (1)$$

still be true, for substituting $90^\circ + m$ for a, we have $\sin (90^\circ + m + b)$ in the place of the first member, which is equal to $\cos (m + b)$†; for the second member by the same substitution we have

$$\sin (90^\circ + m) \cos b + \sin b \cos (90^\circ + m)$$

---

* Tan 45° = cot 45° = $R = 1$.

† By referring to either of the diagrams in which a sine is drawn, it will be evident that $\sin (90^\circ + a)$, a being any arc less than a quadrant, is equal in length to $\sin (90^\circ - a) = \cos a$. Also that $\cos (90^\circ + a) = -\cos (90^\circ - a) = -\sin a$.

but $\sin(90^\circ + m) = \cos m$ and $\cos(90^\circ + m) = -\sin m$, hence equation (1) becomes

$$\cos(m + b) = \cos m \cos b - \sin m \sin b$$

which, since m and b are less than 90°, we know to be true, by Art. 70; hence (1), from which it is derived, is true also.

Assuming (1) to be true with $a > 90^\circ$, which we have just proved, make $b = 90^\circ + m$, and in a similar manner the truth of the formula may be established on the supposition of both a and $b > 90^\circ$.

Afterwards make $a = 180^\circ + m$ and observe that $\sin(180^\circ + m + b) = \sin m + b$ and $\cos(180^\circ + m) = -\cos m$, and you will show that the formula extends to the third quadrant, and so on.

7. Not having been sufficiently advanced in the theory of trigonometrical lines to explain the construction of the tables of sines, tangents, &c., at Art. 38, the explanation is here given.

The diameter of a circle being multiplied by $\pi = 3{\cdot}1415926$ the product is the length of its circumference; this divided by 360 gives us the length of one degree, and this by 60 the length of one minute of the circumference. So small an arc as 1′ may be considered as equal to its sine, without sensible error. Having thus found the sine of 1′ the sines of other arcs may be found by a formula which will now be deduced.

To determine the sine of $(n + 1)\,a$, in terms of $n\,a$, $(n - 1)\,a$ and a

By formulas (8) and (7) of Art. 70.

$$\sin(b + a) = \sin b \cos a + \sin a \cos b \qquad (a)$$

$$\sin(b - a) = \sin b \cos a - \sin a \cos b \qquad (b)$$

Adding these two equations there results

$$\sin(b + a) + \sin(b - a) = 2 \sin b \cos a \qquad (g)$$

Subtracting $\sin(b - a)$ from each member there remains

$$\sin(b + a) = 2 \sin b \cos a - \sin(b - a)$$

Let $b = n\,a$, then the above becomes

$$\sin(n + 1)\,a = 2 \sin n\,a \cos a - \sin(n - 1)\,a \qquad (l)$$

In the last expression let $n = 1$; then $n + 1 = 2$, $n - 1 = 0$

$$\begin{aligned} \therefore\ \sin 2a &= 2 \sin a \cos a - \sin 0 \qquad (m) \\ &= 2 \sin a \cos a \text{ the same result as in } (g), \text{ if } a = b. \end{aligned}$$

Let $n = 2$; then $n + 1 = 3$, $n - 1 = 1$;

$$\begin{aligned} \text{and } \sin 3a &= 2 \sin 2a \cos a - \sin a \\ &= 2 \times 2 \sin a \cos a \times \cos a - \sin a \\ &= 4 \sin a \cos^2 a - \sin a \\ &= 4 \sin a\,(1 - \sin^2 a) - \sin a \\ &= 3 \sin a - 4 \sin^3 a \qquad (n) \end{aligned}$$

Let $n = 3$; then $n + 1 = 4$, $n - 1 = 2$;

And by formula (m):

$$\begin{aligned} \sin 4a &= 2 \times \sin 3a \times \cos a - \sin 2a \\ &= 2\,(3 \sin a - 4 \sin^3 a) \cos a - 2 \sin a \cos a \\ &= (8 \cos^3 a - 4 \cos a) \sin a. \end{aligned}$$

Continuing the same process, we find successively $\sin 5a$, $\sin 6a$, &c.

To determine the cosine of $(n + 1)\,a$, *in terms of* $n\,a$, $(n - 1)\,a$, *and* a.

By formulas (6) and (8) of Art. 70.

$$\cos(b + a) = \cos b \cos a - \sin b \sin a \qquad (c)$$

$$\cos(b - a) = \cos b \cos a + \sin b \sin a \qquad (d)$$

Adding these two equations, we have

$$\cos(b + a) + \cos(b - a) = 2 \cos b \cos a$$

Subtracting $\cos(b - a)$ from each member there results

$$\cos(b + a) = 2 \cos b \cos a - \cos(b - a)$$

If $b = n\,a$, this becomes

$$\cos(n + 1)\,a = 2 \cos n\,a \cos a - \cos(n - 1)\,a \qquad (o)$$

In the formula (o) let $n = 1$; $\therefore n + 1 = 2$, $n - 1 = 0$;

$$\begin{aligned} \text{Then,} \quad \cos 2a &= 2 \cos a \cos a - \cos 0 \\ &= 2 \cos^2 a - 1 \end{aligned}$$

Let $n = 2$, then $n + 1 = 3$, $n - 1 = 1$;

$$\begin{aligned} \text{and } \cos 3a &= 2 \cos 2a \cos a - \cos a \\ &= 2\,(2 \cos^2 a - 1) \cos a - \cos a \\ &= 4 \cos^3 a - 3 \cos a \qquad (p) \end{aligned}$$

Let $n = 3$; then $n + 1 = 4$, $n - 1 = 2$;

$$\begin{aligned} \text{and } \cos 4a &= 2 \cos 3a \cos a - \cos 2a \\ &= 2\,(4 \cos^3 a - 3 \cos a) \cos a - (2 \cos^2 a - 1) \\ &= 8 \cos^4 a - 8 \cos^2 a + 1 \end{aligned}$$

Continuing the same process, we may find, successively, $\cos 5a$, $\cos 6a$, &c.

Making a equal to $1'$ (m) becomes

$$\sin 2' = 2 \sin 1' \cos 1'$$

observing that $\cos 1' = \sqrt{1 - \sin^2 1'}$ (Art. 72); we have thus the value of the sin $2'$. From (n)

$$\sin 3' = 3 \sin 1' - 4 \sin^3 1'$$

and so on. The cosines are calculated in the same manner from (o) or from the sines by the formula

$$\cos = \sqrt{1 - \sin^2}$$

the tangents by the formula (Art. 32),

$$\tan = \frac{R \sin}{\cos}$$

the cotangents by (Art. 34),

$$\cot = \frac{R \cos}{\sin}$$

It is manifest that by continuing the above processes the numerical values of the sines and cosines, &c., of all angles from 1′ up to 90° will be obtained.

The above arithmetical operations are laborious. It is evident that an error in the sine or cosine of the first arcs found will involve errors in the sines, cosines, &c. of all succeeding arcs. Hence the necessity of some check on the computation. For this purpose formulas are employed, called *formulas of verification.*

$\mathrm{Sin}^2\, a + \cos^2 a = 1$

And $2 \sin a \cos a = \sin 2\, a$

Hence, $\sin a = \frac{1}{2} \sqrt{1 + \sin 2\, a} \pm \frac{1}{2} \sqrt{1 - \sin 2\, a}$

$\cos a = \frac{1}{2} \sqrt{1 + \sin 2\, a} \mp \frac{1}{2} \sqrt{1 - \sin 2\, a}$

Now if we suppose $a = 12^\circ\ 30'$

$\sin 12^\circ\ 30' = \frac{1}{2} \sqrt{1 + \sin 25^\circ} \pm \frac{1}{2} \sqrt{1 - \sin 25^\circ}$

$\cos 12^\circ\ 30' = \frac{1}{2} \sqrt{1 + \sin 25^\circ} \mp \frac{1}{2} \sqrt{1 - \sin 25^\circ}$

If the values of the sine and cosine of 12° 30′, and of the sine of 25° obtained by the method previously explained, when substituted in these equations, render the two members identical, the results are supposed correct.

The values of the sine and cosine of 30°, 45°, 60°, &c., may be used as formulas of verification.

The following is known as Euler's formula of verification, which we give without demonstration.

$$\sin a + \sin (36^\circ - a) + \sin (72^\circ + a) = \sin (36^\circ + a) + \sin (72^\circ - a)$$

The following is Legendre's formula.

$$\cos a = \sin (54^\circ + a) + \sin (54^\circ - a) - \sin (18^\circ + a) - \sin (18^\circ - a)$$

To exemplify the latter, make $a = 13^\circ$, then

$$\cos 13^\circ = \sin 67^\circ + \sin 41^\circ - \sin 31^\circ - \sin 5^\circ$$

Substituting for these their values from tab. nat. sines

$$97436 = 92050 + 65606 - 51504 - 08716$$

8. The sines, cosines, &c., may be calculated by series.

To develop sin x *and cos* x *in a series containing the ascending powers of* x, *by the method of undetermined co-efficients.*

The series for sin x must vanish when $x = 0$, therefore it can contain no term independent of x, nor can the even powers of x enter into the series; for suppose,

$$\sin x = \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 + \mathrm{D}\,x^4 + \mathrm{E}\,x^5 +$$

substitute $(-x)$ for x and the above becomes

$$\begin{aligned}
\sin(-x) &= -\mathrm{A}\,x + \mathrm{B}\,x^2 - \mathrm{C}\,x^3 + \mathrm{D}\,x^4 - \mathrm{E}\,x^5 + \\
\text{but } \sin(-x) &= -\sin x = -\mathrm{A}\,x - \mathrm{B}\,x^2 - \mathrm{C}\,x^3 - \mathrm{D}\,x^4 - \mathrm{E}\,x^5 - \\
\therefore \mathrm{B} &= -\mathrm{B},\ \mathrm{D} = -\mathrm{D}, \text{ \&c., absurd unless } \mathrm{B} = 0,\ \mathrm{D} = 0, \text{ \&c.} \\
\therefore \sin x &= \mathrm{A}\,x + \mathrm{C}\,x^3 + \mathrm{E}\,x^5 + \mathrm{G}\,x^7 + \qquad (1)
\end{aligned}$$

Again, the series for cos x must $= 1$ when $x = 0$, its first term must be 1; and it can contain no uneven powers of x, for suppose

$$\begin{aligned}
\cos x &= 1 + \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 + \mathrm{D}\,x^4 + \\
\text{then } \cos(-x) &= 1 - \mathrm{A}\,x + \mathrm{B}\,x - \mathrm{C}\,x^3 + \mathrm{D}\,x^4 - \\
\text{but } \cos(-x) &= \cos x \\
&= 1 + \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 + \mathrm{D}\,x^4 + \\
\because \mathrm{A} &= -\mathrm{A},\ \mathrm{C} = -\mathrm{C}, \qquad \therefore \mathrm{A} = 0,\ \mathrm{C} = 0 \\
\therefore \cos x &= 1 + \mathrm{B}\,x^2 + \mathrm{D}\,x^4 + \qquad (2)
\end{aligned}$$

Adding and subtracting (1) and (2)

$$\begin{aligned}
\cos x + \sin x &= 1 + \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 + \mathrm{D}\,x^4 + \mathrm{E}\,x^5 + \qquad (3) \\
\cos x - \sin x &= 1 - \mathrm{A}\,x + \mathrm{B}\,x^2 - \mathrm{C}\,x^3 + \mathrm{D}\,x^4 - \mathrm{E}\,x^5 + \qquad (4)
\end{aligned}$$

In equation (3) substituting $x + h$ for x, it becomes

$$\begin{aligned}
\cos(x+h) + \sin(x+h) &= 1 + \mathrm{A}\,(x+h) + \mathrm{B}\,(x+h)^2 + \mathrm{C}\,(x+h)^3 + \qquad (5) \\
\text{but } \cos(x+h) + \sin(x+h) &= \cos x \cos h - \sin x \sin h + \sin x \cos h + \cos x \sin h \\
&= \cos h\,(\cos x + \sin x) + \sin h\,(\cos x - \sin x) \\
&= (1 + \mathrm{B}\,h^2 + \mathrm{D}\,h^4 + \ )\ (1 + \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 +) \\
&\quad + (\mathrm{A}\,h + \mathrm{C}\,h^3 + \mathrm{E}\,h^5 + \ )\,(1 - \mathrm{A}\,x + \mathrm{B}\,x^2 - \mathrm{C}\,x^3 +) \\
&= \left.\begin{array}{r}
1 + \mathrm{A}\,x + \mathrm{E}\,x^2 + \mathrm{C}\,x^3 \quad + \\
+ \mathrm{A}\,h - \mathrm{A}^2 xh + \mathrm{A\,B}\,x^2 h + \\
+ \mathrm{B}\,h^2 + \mathrm{A\,B}\,xh^2 + \\
+ \mathrm{C}\,h^3 \quad + \\
+
\end{array}\right\} \qquad (6)
\end{aligned}$$

Comparing (5) and (6) we have

$$\left.\begin{array}{r}
1 + \mathrm{A}\,x + \mathrm{B}\,x^2 + \mathrm{C}\,x^3 \quad + \\
+ \mathrm{A}\,h + 2\mathrm{B}\,xh + 3\mathrm{C}\,x^2 h \ + \\
+ \mathrm{B}\,h^2 + 3\mathrm{C}\,xh^2 + \\
+ \mathrm{C}\,h^3 \quad +
\end{array}\right\} = \left.\begin{array}{r}
1 + \mathrm{A}\,x + \mathrm{B}\,x^2 \quad + \mathrm{C}\,x^3 \quad + \\
+ \mathrm{A}\,h - \mathrm{A\,A}\,xh + \mathrm{A\,B}\,x^2 h - \\
+ \mathrm{B}\,h^2 \quad + \mathrm{A\,B}\,xh^2 + \\
+ \mathrm{C}\,h^3 \quad - \\
+
\end{array}\right\}$$

and equating the coefficients of the terms containing the same powers of $x\ h$,

$$2\mathrm{B} = -\mathrm{A}\,\mathrm{A}\,; \therefore\ \mathrm{B} = -\frac{\mathrm{A}\,\mathrm{A}}{2} = -\frac{\mathrm{A}^2}{1\cdot 2}$$

$$3\mathrm{C} = \mathrm{A}\,\mathrm{B} \qquad \mathrm{C} = \frac{\mathrm{A}\,\mathrm{B}}{3} = -\frac{\mathrm{A}^3}{1\cdot 2\cdot 3}$$

$$4\mathrm{D} = -\mathrm{A}\,\mathrm{C} \qquad \mathrm{D} = -\frac{\mathrm{A}\,\mathrm{C}}{4} = +\frac{\mathrm{A}^4}{1\cdot 2\cdot 3\cdot 4}$$

$$5\mathrm{E} = \mathrm{A}\,\mathrm{D} \qquad \mathrm{E} = \frac{\mathrm{A}\,\mathrm{D}}{5} = +\frac{\mathrm{A}^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

hence $\sin x = \mathrm{A}\,x - \frac{\mathrm{A}^3}{1\cdot 2\cdot 3}x^3 + \frac{\mathrm{A}^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \frac{\mathrm{A}^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 +$

$$\cos x = 1 - \frac{\mathrm{A}^2}{1\cdot 2}x^2 + \frac{\mathrm{A}^4}{1\cdot 2\cdot 3\cdot 4}x^4 - \frac{\mathrm{A}^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6 +$$

It remains now to determine the value of A. For this purpose we have

$$\sin x = \mathrm{A}\,x - \frac{\mathrm{A}^3}{1\cdot 2\cdot 3}x^3 + \frac{\mathrm{A}^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 -$$

$$= \mathrm{A}\,x\left(1 - \frac{\mathrm{A}^2}{1\cdot 2\cdot 3}x^2 + \frac{\mathrm{A}^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^4 - \right)$$

If x be very small, the first member $\sin x = x$ and the terms in the parenthesis after the first vanish. Hence $x = \mathrm{A}\,x \ \therefore\ \mathrm{A} = 1$, and

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3} + \frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{x^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \qquad (1)$$

$$\cos x = 1 - \frac{x^2}{1\cdot 2} + \frac{x^4}{1\cdot 2\cdot 3\cdot 4} - \frac{x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \qquad (2)$$

Now, the length of an arc of one degree is so small that if x be equal to this, the third term of the above series will contain no significant figure in the first ten places of decimals.

Retaining therefore only the first two terms of (1) we have, when x is small,

$$\sin x = x - \frac{x^3}{1\cdot 2\cdot 3} = x\,(1 - \frac{x^2}{2\cdot 3}) = x\left\{1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4}\right\}^{\frac{1}{3}} \text{ nearly;}$$

that is, the quantity within the brackets being $= \cos x$, by (2)

$$\sin x = x \cos \tfrac{1}{3}\, x\,;$$

therefore introducing radius to render the expression homogeneous, we have

$$\log.\, \sin x = \log.\, x - \tfrac{1}{3}\,(10 - \log.\, \cos x) \qquad (3)$$

Suppose the arc x to contain n seconds, then

$$x = \frac{n\cdot\pi}{180 \times 60 \times 60}$$

---

* Dividing (1) by (2) we have $\tan x = x + \frac{1}{3}x^3 \ldots$

Introducing radius and applying logarithms

$$\begin{aligned}\text{log. } x &= \text{log. } n + \text{log. } 3{\cdot}14159, \text{ \&c.} + 10 - \text{log. } \overline{180 \times 60^2} \\ &= \text{log. } n + 4{\cdot}68558\,;\end{aligned}$$

substituting this value of log. x in (3) it becomes

$$\text{log. sin } x = \text{log. } n + 4{\cdot}68558 - \tfrac{1}{3} \text{ ar. comp. log. cos } x. \qquad (4)$$

Hence this rule. To the logarithm of the arc reduced into seconds add the constant 4•68558, and from the sum subtract one-third of the arithmetical complement of the log. cosine; the remainder will be the logarithmic sine of the given arc.

9. *To find the logarithmic tangent of a very small arc.*

By the last article

$$\sin x = x \cos \tfrac{1}{3} x \;\therefore\; \frac{\sin x}{\cos x} = \tan x = \frac{x}{\cos \tfrac{2}{3} x} \; *$$

Introducing the radius and applying logs.

$$\text{log. tan } x = \text{log. } x + \tfrac{2}{3}\,(10 - \text{log. cos } x)$$

The second member of this equation may be formed from the second member of (3) in the last article, by adding the arithmetical complement of the log. cos x; therefore from (4)

$$\text{log. tan } x = \text{log. } n + 4{\cdot}68558 + \tfrac{2}{3} \text{ arith. comp. log. cos } x \qquad (5)$$

hence this rule. Add the logarithm of the arc reduced to seconds, the constant 4•68558, and two-thirds of the arithmetical complement of the log. cosine, the sum is the log. tangent required.

10. *To find a small arc from its log. sine.*

From (4) Art. 8, Appendix I.,

$$\begin{aligned}\text{log. } n &= \text{log. sin } x - 4{\cdot}68558 + \tfrac{1}{3} \text{ arith. comp. log. cos } x \\ &= \text{log. sin } x + 5{\cdot}31442 + \tfrac{1}{3} \text{ arith comp. log. cos } x - 10\end{aligned}$$

Hence the rule is this. Add the log. sine of the arc, the constant 5•31442, and ⅓ of the arithmetical complement of the log. cosine; subtract 10 from the index of the sum, and the remainder will be the logarithm of the number of seconds in the arc.

To find a small arc from its log. tangent.

From (5) last art.

$$\begin{aligned}\text{log. } n &= \text{log. tan } x - 4{\cdot}68558 - \tfrac{2}{3} \text{ arith. comp. log. cos } x \\ &= \text{log. tan } x + 5{\cdot}31442 - \tfrac{2}{3} \text{ arith. comp. log. cos } x - 10\,;\end{aligned}$$

Rule.—Add the log. tangent of the arc, the constant 5•31442, and subtract ⅔ of the arithmetical complement of the log. cosine, reject 10 from the index, and the result will be the logarithm of the arc in seconds.

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* If we substitute for cos x in the last fraction its value $\dfrac{1}{\sqrt{1 + \tan^2 x}}$ we may easily deduce $x = \tan x - \tfrac{1}{3} \tan^3 x$, &c.

11. The values of a, which satisfy the equation $\sin a = 0$ are $0, \pm \pi$,* $\pm 2\pi, \pm 3\pi \ldots . \pm n\pi$

Consequently the series which is the development of $\sin a$ must be divisible by a, $a - \pi, a + \pi, a - 2\pi, a + 2\pi. \ldots$ (Alg. Art. 238, Prop. II.)

If therefore k be a constant whose value is afterwards to be determined, we have

$$\sin a = \pm k\, a\, (\pi - a)\, (\pi + a)\, (2\pi - a)\, (2\pi + a) \ldots$$

$$= \pm k\, a\, \pi\, (1 - \frac{a}{\pi})\, \pi\, (1 + \frac{a}{\pi})\, 2\pi\, (1 - \frac{a}{2\pi})\, 2\pi\, (1 + \frac{a}{2\pi}) \ldots$$

$$= \pm k\, a\, \pi^2 \cdot 2^2 \pi^2 \cdot 3^2 \pi^2 \ldots (1 - \frac{a^2}{\pi^2})\ (1 - \frac{a^2}{2^2 \pi^2}) \ldots$$

$$\therefore \frac{\sin a}{a} = \pm k\, \pi^2 \cdot 2^2 \pi^2 \cdot 3^2 \pi^2 \ldots (1 - \frac{a^2}{\pi^2})\ (1 - \frac{a^2}{2^2 \pi^2}) \ldots$$

If $a = 0, \frac{\sin a}{a} = 1 \;\therefore\; 1 = \pm k\, \pi^2 \cdot 2^2 \pi^2 \cdot 3^2 \pi^2 \ldots$

$$\therefore \sin a = a\, (1 - \frac{a^2}{\pi^2})\ (1 - \frac{a^2}{2^2 \pi^2})\ (1 - \frac{a^2}{3^2 \pi^2}) \ldots \qquad (1)$$

In a similar manner may be derived

$$\cos a = (1 - \frac{2^2 a^2}{\pi^2})\ (1 - \frac{2^2 a^2}{3^2 \pi^2})\ (1 - \frac{2^2 a^2}{5^2 \pi^2}) \ldots \qquad (2)$$

If $a = \frac{1}{2}\pi$, (1) becomes

$$1 = \frac{\pi}{2}\, (1 - \frac{1}{2^2})\ (1 - \frac{1}{4^2})\ (1 - \frac{1}{6^2})\ (1 - \frac{1}{8^2}) \ldots$$

$$= \frac{\pi}{2}\ \frac{2^2 - 1}{2^2} \cdot \frac{4^2 - 1}{4^2} \cdot \frac{6^2 - 1}{6^2} \cdot \frac{8^2 - 1}{8^2}$$

$$= \frac{\pi}{2}\ \frac{(2 - 1)\ (2 + 1)}{2^2} \cdot \frac{(4 - 1)\ (4 + 1)}{4^2} \cdot \frac{(6 - 1)\ (6 + 1)}{6^2} \ldots$$

$$\therefore \pi = 2 \cdot \frac{2^2}{1 \cdot 3} \cdot \frac{4^2}{3 \cdot 5} \cdot \frac{6^2}{5 \cdot 7} \cdot \frac{8^2}{7 \cdot 9} \ldots \dagger$$

Applying logarithms to (1) we have, making $a = \frac{m}{n}\, \frac{\pi}{2}$

$$\log.\, \sin \frac{m}{n}\, \frac{\pi}{2} = \log.\, (\frac{m}{n}\, \frac{\pi}{2}) + \log.\, (1 - \frac{m^2}{2^2\, n^2}) + \log.\, (1 - \frac{m^2}{4^2\, n^4})$$
$$+ \log.\, (1 - \frac{m^2}{6^2\, n^6}) + \ldots$$

---

* π being 180°, or the semi-circumference whose radius is 1.

† This is the theorem of Wallis for determining π. Its successive factors approximate each more and more to 1.

Developing log. $\left(1-\frac{m^2}{4^2\,n^2}\right)$, log. $\left(1-\frac{m^2}{6^2\,n^2}\right)$ &c. Alg. Art. 224.

$$\log.\sin\frac{m}{n}\frac{\pi}{2}=\log.\frac{m}{n}\frac{\pi}{2}+\log.\left(\frac{2^2\,n^2-m^2}{2^2\,n^2}\right)$$

$$-\mathrm{M}\left(\frac{m^2}{4^2\,n^2}+\tfrac{1}{2}\,\frac{m^4}{4^4\,n^4}+\tfrac{1}{3}\,\frac{m^6}{4^6\,n^6}\cdots\right)$$

$$-\mathrm{M}\left(\frac{m^2}{6^2\,n^2}+\tfrac{1}{2}\,\frac{m^4}{6^4\,n^4}+\tfrac{1}{3}\,\frac{m^6}{6^6\,n^6}\cdots\right)$$

$$-\mathrm{M}\left(\frac{m^2}{8^2\,n^2}+\tfrac{1}{2}\,\frac{m^4}{8^4\,n^4}+\tfrac{1}{3}\,\frac{m^6}{8^6\,n^6}\cdots\right)$$

But,

$$\log.\left(\frac{m}{n}\frac{\pi}{2}\right)=\log.m+\log.\pi-\log.n-\log.2$$

And

$$\log.\frac{2^2\,n^2-m^2}{2^2\,n^2}=\log.\left\{(2n+m)\,(2n-m)\right.-\log.2^2\,n^2$$

$$=\log.(2n+m)+\log.(2n-m)-2\,(\log.2+\log.n)$$

$$\therefore\ \log.\sin\frac{m}{n}\frac{\pi}{2}=\log.m+\log.(2n+m)+\log.(2n-m)+\log\pi$$

$$-3\,(\log.n+\log.2.)$$

$$-\mathrm{M}^*\left\{\begin{array}{l}\left(\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}\cdots\right)\frac{m^2}{n^2}\\+\tfrac{1}{2}\left(\frac{1}{4^4}+\frac{1}{6^4}+\frac{1}{8^4}\cdots\right)\frac{m^4}{n^4}\\+\tfrac{1}{3}\left(\frac{1}{4^6}+\frac{1}{6^6}+\frac{1}{8^6}\ldots\right)\frac{m^6}{n^6}\\+\ \&c.\end{array}\right\}$$

The above form serves to compute a logarithmic sine at once, without first computing the natural sine.†

From (2) in a similar manner may be derived the following formula for calculating logarithmic cosines.

$$\log.\cos\left(\frac{m}{n}\frac{\pi}{2}\right)=\log.(n+m)+\log.(n-m)-2\log.n$$

$$-\mathrm{M}\left\{\begin{array}{l}\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}\cdots\right)\frac{m^2}{n^2}\\+\tfrac{1}{2}\left(\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}\cdots\right)\frac{m^4}{n^4}\\+\tfrac{1}{3}\left(\frac{1}{3^6}+\frac{1}{5^6}+\frac{1}{7^6}\cdots\right)\frac{m^6}{n^6}\\+\ \&c.\end{array}\right\}$$

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[*] M is the modulus of the system, equal for the common to ·43429448.

[†] $\frac{m}{n}$ expresses what fraction of a quadrant the arc is.

12. Adding and subtracting (3) and (7) of Art. 70, and adding and subtracting (6) and (8) of the same article, the following formulas are obtained.

$$\left.\begin{aligned} \sin(a+b)+\sin(a-b) &= 2\sin a\cos b \\ \sin(a+b)-\sin(a-b) &= 2\sin b\cos a \\ \cos(a+b)+\cos(a-b) &= 2\cos a\cos b \\ \cos(a+b)-\cos(a-b) &= -2\sin a\sin b \end{aligned}\right\} \qquad (a)$$

The following two are identical equations

$$a = \frac{a+b}{2} + \frac{a-b}{2}$$

$$b = \frac{a+b}{2} - \frac{a-b}{2}$$

$$\therefore \sin a = \sin\left\{\frac{a+b}{2} + \frac{a-b}{2}\right\}$$

$$= \sin\frac{a+b}{2}\cos\frac{a-b}{2} + \sin\frac{a-b}{2}\cos\frac{a+b}{2} \qquad (1)$$

$$\sin b = \sin\left\{\frac{a+b}{2} - \frac{a-b}{2}\right\}$$

$$= \sin\frac{a+b}{2}\cos\frac{a-b}{2} - \sin\frac{a-b}{2}\cos\frac{a+b}{2} \qquad (2)$$

$$\cos a = \cos\left\{\frac{a+b}{2} + \frac{a-b}{2}\right\}$$

$$= \cos\frac{a+b}{2}\cos\frac{a-b}{2} - \sin\frac{a+b}{2}\sin\frac{a-b}{2} \qquad (3)$$

$$\cos b = \cos\left\{\frac{a+b}{2} - \frac{a-b}{2}\right\}$$

$$= \cos\frac{a+b}{2}\cos\frac{a-b}{2} + \sin\frac{a+b}{2}\sin\frac{a-b}{2} \qquad (4)$$

Adding together (1) and (2) there results

$$\sin a + \sin b = 2\sin\frac{a+b}{2}\cos\frac{a-b}{2} \qquad (5)$$

Subtracting (2) from (1),

$$\sin a - \sin b = 2\sin\frac{a-b}{2}\cos\frac{a+b}{2} \qquad (6)$$

Add together (3) and (4),

$$\cos a + \cos b = 2\cos\frac{a+b}{2}\cos\frac{a-b}{2} \qquad (7)$$

Subtract (4) from (3),

$$\cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2} \qquad (8)$$

These formulas might have been immediately deduced from the group (*q*), by changing $a+b$ into a, $a-b$ into b, a into $\frac{a+b}{2}$ b into $\frac{a-b}{2}$

13. The following are various forms of expressing the principles deduced from Art. 14 to Art. 31 inclusive.

$\sin 0 = 0$
$\cos 0 = 1$
$\tan 0 = 0$
$\cot 0 = \infty$
$\sec 0 = 1$
$\operatorname{cosec} 0 = \infty$
$\sin (90^\circ - a) = \cos a$
$\cos (90^\circ - a) = \sin a$
$\tan (90^\circ - a) = \cot a$
$\cot (90^\circ - a) = \tan a$
$\sec (90^\circ - a) = \operatorname{cosec} a$
$\operatorname{cosec} (90^\circ - a) = \sec a$
$\sin 90^\circ = 1$
$\cos 90^\circ = 0$
$\tan 90^\circ = \infty$
$\cot 90^\circ = 0$
$\sec 90^\circ = \infty$
$\operatorname{cosec} 90^\circ = 1$
$\sin (90^\circ + a) = \cos a$
$\cos (90^\circ + a) = -\sin a$
$\tan (90^\circ + a) = -\cot a$
$\cot (90^\circ + a) = -\tan a$
$\sec (90^\circ + a) = -\operatorname{cosec} a$
$\operatorname{cosec} (90^\circ + a) = \sec a$
$\sin (180^\circ - a) = \sin a$
$\cos (180^\circ - a) = -\cos a$
$\tan (180^\circ - a) = -\tan a$
$\cot (180^\circ - a) = -\cot a$
$\sec (180^\circ - a) = -\sec a$
$\operatorname{cosec} (180^\circ - a) = \operatorname{cosec} a$
$\sin 180^\circ = 0$
$\cos 180^\circ = -1$
$\tan 180^\circ = 0$
$\cot 180^\circ = -\infty$
$\sec 180^\circ = -1$
$\operatorname{cosec} 180^\circ = 0$

$\sin (180^\circ + a) = -\sin a$
$\cos (180^\circ + a) = -\cos a$
$\tan (180^\circ + a) = \tan a$
$\cot (180^\circ + a) = \cot a$
$\sec (180^\circ + a) = -\sec a$
$\operatorname{cosec} (180^\circ + a) = -\operatorname{cosec} a$
$\sin (270^\circ - a) = -\cos a$
$\cos (270^\circ - a) = -\sin a$
$\tan (270^\circ - a) = \cot a$
$\cot (270^\circ - a) = \tan a$
$\sec (270^\circ - a) = -\operatorname{cosec} a$
$\operatorname{cosec} (270^\circ - a) = -\sec a$
$\sin 270^\circ = -1$
$\cos 270^\circ = 0$
$\tan 270^\circ = -\infty$
$\cot 270^\circ = 0$
$\sec 270^\circ = -\infty$
$\operatorname{cosec} 270^\circ = -1$
$\sin (270^\circ + a) = -\cos a$
$\cos (270^\circ + a) = \sin a$
$\tan (270^\circ + a) = -\cot a$
$\cot (270^\circ + a) = -\tan a$
$\sec (270^\circ + a) = \operatorname{cosec} a$
$\operatorname{cosec} (270^\circ + a) = -\sec a$
$\sin (360^\circ - a) = -\sin a$
$\cos (360^\circ - a) = \cos a$
$\tan (360^\circ - a) = -\tan a$
$\cot (360^\circ - a) = -\cot a$
$\sec (360^\circ - a) = \sec a$
$\operatorname{cosec} (360^\circ - a) = -\operatorname{cosec} a$
$\sin 360^\circ = 0$
$\cos 360^\circ = 1$
$\tan 360^\circ = 0$
$\cot 360^\circ = \infty$
$\sec 360^\circ = 1$
$\operatorname{cosec} 360^\circ = \infty$*

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* All the infinities in the above table should perhaps strictly have the doubtful sign, since they are the transition value.

$$\sin a = (\sin n\,360^\circ + a) = \sin (2n\,180^\circ + a)$$

$$\sin a = \sin \left\{ (2n+1)\,180^\circ - a \right\}$$

$$\sin a = -\sin \left\{ (2n+1)\,180^\circ + a \right\}$$

$$\sin a = -\sin (2n \cdot 180^\circ - a)$$

$$\cos a = \cos (2n \cdot 180^\circ + a) \text{ or } -\cos \left\{ (2n+1)\,180^\circ - a \right\}$$

$$\text{or } -\cos \left\{ (2n+1)\,180^\circ + a \right\} \text{ or } \cos (2n \cdot 180^\circ - a)$$

$$\tan a = \tan (2n \cdot 180^\circ + a) \text{ or } -\tan \left\{ (2n+1)\,180^\circ - a \right\}$$

$$\text{or } \tan \left\{ (2n+1)\,180^\circ + a \right\} \text{ or } -\tan (2n \cdot 180^\circ - a)$$

$$\sec a = \sec (2n \cdot 180^\circ + a) \text{ or } -\sec \left\{ (2n+1)\,180^\circ - a \right\}$$

$$\text{or } -\sec \left\{ (2n+1)\,180^\circ + a \right\} \text{ or } \sec (2n \cdot 180^\circ - a)$$

14. The following are the most useful general relations of arcs or angles deduced in the preceding pages, or deducible from the formulas which they contain.

TABLE II.

(1.) $\sin (a \pm b) = \sin a \cos b \pm \sin b \cos a$

(2.) $\cos (a \pm b) = \cos a \cos b \mp \sin a \sin b$

(3.) $\tan (a \pm b) = \dfrac{\tan a \pm \tan b}{1 \mp \tan a \tan b}$

(4.) $\sin 2a = 2 \sin a \cos a$ also $= \dfrac{2 \tan a}{1 + \tan^2 a} = \dfrac{2\sqrt{\sec^2 a - 1}}{\sec^2 a} = \dfrac{2 \cot a}{1 + \cot^2 a} = \dfrac{2\sqrt{\operatorname{cosec}^2 a - 1}}{\operatorname{cosec}^2 a}$

(5.) $\cos 2a = \cos^2 a - \sin^2 a = 2 \cos^2 a - 1 = 1 - 2 \sin^2 a = \dfrac{1 - \tan^2 a}{1 + \tan^2 a} = \dfrac{2 - \sec^2 a}{\sec^2 a} = \dfrac{\cot^2 a - 1}{\cot^2 a + 1} = \dfrac{\operatorname{cosec}^2 a - 2}{\operatorname{cosec}^2 a} = 1 - 2\,(2 \operatorname{vers} a - \operatorname{vers}^2 a)$

(6.) $\tan 2a = \dfrac{2 \tan a}{1 - \tan^2 a}$

(7.) $\sin \dfrac{a}{2} = \sqrt{\dfrac{1 - \cos a}{2}}$

(8.) $\cos \dfrac{a}{2} = \sqrt{\dfrac{1 + \cos a}{2}}$

(9.) $\tan \dfrac{a}{2} = \sqrt{\dfrac{1 - \cos a}{1 + \cos a}} = \dfrac{1 - \cos a}{\sin a} = \dfrac{\sin a}{1 + \cos a}$

(10.) $\sin a = 2 \sin \frac{a}{2} \cos \frac{a}{2}$

(11.) $\sin^2 a = -\frac{1}{2} (\cos 2a - 1)$ (See Art. 18, App. I.)

(12.) $\sin^3 a = -\frac{1}{2^2} (\sin 3a - 3 \sin a)$

(13.) $\sin^4 a = -\frac{1}{2^3} (\cos 4a - 4 \cos 2a + \frac{1}{2} \frac{4 \cdot 3}{1 \cdot 2})$

(14.) $\sin^5 a = \frac{1}{2^4} (\sin 5a - 5 \sin 3a + \frac{5 \cdot 4}{1 \cdot 2} \sin a)$

(15.) $\sin^6 a = -\frac{1}{2^5} (\cos 6a - 6 \cos 4a + \frac{6 \cdot 5}{1 \cdot 2} \cos 2a - \frac{1}{2} \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3})$

(16.) $\cos^2 a = \frac{1}{2} (\cos 2a + 1)$

(17.) $\cos^3 a = \frac{1}{2^2} (\cos 3a + 3 \cos a)$

(18.) $\cos^4 a = \frac{1}{2^3} (\cos 4a + 4 \cos 2a + \frac{1}{2} \frac{4 \cdot 3}{1 \cdot 2})$

(19.) $\cos^5 a = \frac{1}{2^4} (\cos 5a + 5 \cos 3a + \frac{5 \cdot 4}{1 \cdot 2} \cos a)$

(20.) $\cos^6 a = \frac{1}{2^5} (\cos 6a + 6 \cos 4a + \frac{6 \cdot 5}{1 \cdot 2} \cos 2a + \frac{1}{2} \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3})$

(21.) $\sin (n + 1) a = 2 \sin na \cos a - \sin (n - 1) a$

(22.) $\cos (n + 1) a = 2 \cos na \cos a - \cos (n - 1) a$

(23.) $\sin a + \sin b = 2 \sin \frac{a + b}{2} \cos \frac{a - b}{2}$

(24.) $\sin a - \sin b = 2 \sin \frac{a - b}{2} \cos \frac{a + b}{2}$

(25.) $\cos a + \cos b = 2 \cos \frac{a + b}{2} \cos \frac{a - b}{2}$

(26.) $\cos a - \cos b = -2 \sin \frac{a + b}{2} \sin \frac{a - b}{2}$

(27.) $\frac{\sin a + \sin b}{\sin a - \sin b} = \frac{\tan \frac{a + b}{2}}{\tan \frac{a - b}{2}}$

(28.) $\frac{\sin a \pm \sin b}{\cos a + \cos b} = \tan \frac{1}{2} (a \pm b)$

(29.) $\frac{\sin a \pm \sin b}{\cos b - \cos a} = \cot \frac{1}{2} (a \mp b)$

(30.) $\sin (a + b) + \sin (a - b) = 2 \sin a \cos b$

(31.) $\sin (a - b) - \sin (a - b) = 2 \sin b \cos a$

(32.) $\cos (a + b) + \cos (a - b) = 2 \cos a \cos b$

(33.) $\cos (a + b) - \cos (a - b) = -2 \sin a \sin b$

(34.) $2 \cos \frac{45^\circ}{2^n} = \sqrt{2 \sqrt{2 \sqrt{2}}}$ &c. to $n + 1$ radicals

(35.) $\cos 36^\circ = \sin 54^\circ = \frac{1}{4}(1 + \sqrt{5})$

(36.) $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$

(37.) $\tan 45^\circ = \cot 45^\circ = 1$

(38.) $\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$

(39.) $\cos 30^\circ = \sin 60^\circ = \frac{\sqrt{3}}{2}$

(40.) $\tan 30^\circ = \cot 60^\circ = \frac{1}{\sqrt{3}}$

(41.) $\cot 30^\circ = \tan 60^\circ = \sqrt{3}$

The formulas of Trigonometry may be varied to almost any extent, and the same quantity expressed in many different ways.

15. The following, of less frequent occurrence, may be readily deduced from the above.

(42.) $\left\{\begin{matrix} \sin(45^\circ \pm a) \\ \cos(45^\circ \mp a \end{matrix}\right\} \begin{matrix} = \\ = \end{matrix} \frac{\cos a \pm \sin a}{\sqrt{2}}$

(43.) $\tan(45^\circ \pm a) = \frac{1 \pm \tan a}{1 \mp \tan a}$

(44.) $\tan^2 (45^\circ \pm \frac{a}{2}) = \frac{1 \pm \sin a}{1 \mp \sin a}$

(45.) $\tan(45^\circ \pm \frac{a}{2}) = \frac{1 \pm \sin a}{\cos a} = \frac{\cos a}{1 \mp \sin a}$

(46.) $\frac{\sin(a + b)}{\sin(a - b)} = \frac{\tan a + \tan b}{\tan a - \tan b} = \frac{\cot b + \cot a}{\cot b - \cot a}$

(47.) $\frac{(\cos a + b)}{\cos(a - b)} = \frac{\cot b - \tan a}{\cot b + \tan a} = \frac{\cot a - \tan b}{\cot a + \tan a}$

(48.) $\sin a = \frac{\varepsilon^{a\sqrt{-1}} - \varepsilon^{-a\sqrt{-1}}}{2\sqrt{-1}}$ [*]

(49.) $\cos a = \frac{\varepsilon^{a\sqrt{-1}} + \varepsilon^{-a\sqrt{-1}}}{2}$

(50.) $\varepsilon^{a\sqrt{-1}} = \cos a + \sqrt{-1} \sin a$

(51.) $\varepsilon^{-a\sqrt{-1}} = \cos a - \sqrt{-1}\ \text{in}\ a$

(52.) $\frac{\cos a + \cos b}{\cos a - \cos b} = -\cot \frac{a + b}{2} \cot \frac{a - b}{2}$

(53.) $\tan a + \tan b = \frac{\sin(a + b)}{\cos a \cos b}$

(54.) $\cot a + \cot b = \frac{\sin(a + b)}{\sin a \sin b}$

---

[*] ε is the base of the Naperian system of logarithms. For the demonstration of the first two of these four formulas see next article.

(55.) $\tan a - \tan b = \dfrac{\sin(a-b)}{\cos a \cos b}$

(56.) $\cot a - \cot b = -\dfrac{\sin(a-b)}{\sin a \sin b}$

(57.) $\cos a + \sin a = \sqrt{1 + \sin 2a}$

(58.) $\cos a - \sin a = \sqrt{1 - \sin 2a}$

If a be less than 45° then

(59.) $\cos a = \frac{1}{2}\left\{\sqrt{1+\sin 2a} + \sqrt{1-\sin 2a}\right\}$

(60.) $\sin a = \frac{1}{2}\left\{\sqrt{1+\sin 2a} - \sqrt{1-\sin 2a}\right\}$

(61.) $\dfrac{\tan a + \tan b}{\tan a - \tan b} = \dfrac{\sin(a+b)}{\sin(a-b)}$

(62.) $\dfrac{\cos a + \sin a}{\cos a - \sin a} = \sec 2a + \tan 2a$

(63.) $\sin(a+b)\sin(a-b) = \sin^2 a - \sin^2 b = \cos^2 b - \cos^2 a$

(64.) $\cos(a+b)\cos(a-b) = \cos^2 a - \sin^2 b = \cos^2 b - \sin^2 a$

(65.) $\sin 3a = 3\sin a - 4\sin^3 a$

(66.) $\cos 3a = 4\cos^3 a - 3\cos a$

(67.) $\cos 2a = \dfrac{1}{1 + \tan 2a \tan a}$

(68.) $\tan^2 a - \tan^2 b = \dfrac{\sin(a+b)\sin(a-b)}{\cos^2 a \cos^2 b}$

(69.) $\cot^2 a - \cot^2 b = -\dfrac{\sin(a+b)\sin(a-b)}{\sin^2 a \sin^2 b}$

(70.) $\dfrac{1-\cos a}{1+\cos a} = \tan^2 \frac{1}{2} a$

(71.) $\dfrac{\sin a}{1+\cos a} = \tan \frac{1}{2} a$

(72.) $\dfrac{1+\cos a}{\sin a} = \cot \frac{1}{2} a$

(73.) $\dfrac{\sin a}{1-\cos a} = \cot \frac{1}{2} a$

(74.) $\dfrac{1-\cos a}{\sin a} = \tan \frac{1}{2} a$

(75.) $\sin a = a - \dfrac{a^3}{1{\cdot}2{\cdot}3} + \dfrac{a^5}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5} - \dfrac{a^7}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6{\cdot}7} +$ &c.

(76.) $\cos a = 1 - \dfrac{a^2}{1{\cdot}2} + \dfrac{a^4}{1{\cdot}2{\cdot}3{\cdot}4} - \dfrac{a^6}{1{\cdot}2{\cdot}3{\cdot}4{\cdot}5{\cdot}6} +$ &c.

(77.) $\tan a = a + \frac{1}{3} a^3 + \frac{2}{15} a^5 + \frac{17}{315} a^7 +$ &c.

(78.) $z = \text{arc}(\sin = y)$ or $\sin^{-1} y$[*] $= y + \frac{1}{6} y^3 + \frac{3}{40} y^5 +$ &c.

(79.) $z = \cos^{-1} x = (1-x) + \frac{1}{6}(1-x^3) + \frac{3}{40}(1-x^5) +$ &c.

(80.) $z = \tan^{-1} t = t - \frac{1}{3} t^3 + \frac{1}{5} t^5$ &c.

15. The following will be found a useful table of reference.

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[*] Arc $(\sin = y)$ or $\sin^{-1} y$ are read the arc whose sin is y.

TABLE OF THE MOST USEFUL ANALYTICAL VALUES OF SIN a, COS a, TAN a.

VALUES OF SIN a.	VALUES OF COS a.	VALUES OF TAN a.
1. $\cos a \tan a$	16. $\dfrac{\sin a}{\tan a}$	31. $\dfrac{\sin a}{\cos a}$
2. $\dfrac{\cos a}{\cot a}$	17. $\sin a \cot a$	32. $\dfrac{1}{\cot a}$
3. $\sqrt{1-\cos^2 a}$	18. $\sqrt{1-\sin^2 a}$	33. $\sqrt{\dfrac{1}{\cos^2 a}-1}$
4. $\dfrac{1}{\sqrt{1+\cot^2 a}}$	19. $\dfrac{1}{\sqrt{1+\tan^2 a}}$	34. $\dfrac{\sin a}{\sqrt{1-\sin^2 a}}$
5. $\dfrac{\tan a}{\sqrt{1+\tan^2 a}}$	20. $\dfrac{\cot a}{\sqrt{1+\cot^2 a}}$	35. $\dfrac{\sqrt{1-\cos^2 a}}{\cos a}$
6. $2 \sin\frac{a}{2} \cos\frac{a}{2}$	21. $\cos^2\frac{a}{2}-\sin^2\frac{a}{2}$	36. $\dfrac{2\tan\frac{a}{2}}{1-\tan^2\frac{a}{2}}$
7. $\sqrt{\dfrac{1-\cos 2a}{2}}$	22. $1-2\sin^2\frac{a}{2}$	37. $\dfrac{2\cot\frac{a}{2}}{\cot^2\frac{a}{2}-1}$
8. $\dfrac{2\tan\frac{a}{2}}{1+\tan^2\frac{a}{2}}$	23. $2\cos^2\frac{a}{2}-1$	38. $\dfrac{2}{\cot\frac{a}{2}-\tan\frac{a}{2}}$
9. $\dfrac{2}{\cot\frac{a}{2}+\tan\frac{a}{2}}$	24. $\sqrt{\dfrac{1+\cos 2a}{2}}$	39. $\cot a - 2\cot 2a$
10. $\dfrac{(\sin 30^\circ+a)-\sin(30^\circ-a)}{\sqrt{3}}$	25. $\dfrac{1-\tan^2\frac{a}{2}}{1+\tan^2\frac{a}{2}}$	40. $\dfrac{1-\cos 2a}{\sin 2a}$
11. $2\sin^2(45^\circ+\frac{a}{2})-1$	26. $\dfrac{\cot\frac{a}{2}-\tan\frac{a}{2}}{\cot\frac{a}{2}+\tan\frac{a}{2}}$	41. $\dfrac{\sin 2a}{1+\cos 2a}$
12. $1-2\sin^2(45^\circ-\frac{a}{2})$	27. $\dfrac{1}{1+\tan a\tan\frac{a}{2}}$	42. $\sqrt{\dfrac{1-\cos 2a}{1+\cos 2a}}$
13. $\dfrac{1-\tan^2(45^\circ-\frac{a}{2})}{1+\tan^2(45^\circ-\frac{a}{2})}$	28. $\dfrac{2}{\tan(45^\circ+\frac{a}{2})+\cot(45^\circ+\frac{a}{2})}$	43. $\dfrac{\tan(45^\circ+\frac{a}{2})-\tan(45^\circ-\frac{a}{2})}{2}$
14. $\dfrac{\tan(45^\circ+\frac{a}{2})-\tan(45^\circ-\frac{a}{2})}{\tan(45^\circ+\frac{a}{2})+\tan(45^\circ-\frac{a}{2})}$	29. $2\cos(45^\circ+\frac{a}{2})\cos(45^\circ-\frac{a}{2})$	
15. $\sin(60^\circ+a)-\sin(60^\circ-a)$	30. $\cos(60^\circ+a)+\cos(60^\circ-a)$	

16. *To develop* A^x *in a series of terms arranged according to the ascending powers of* X.

$$a^x = \left\{ 1 + (a-1) \right\}^x = \text{by the binomial theorem}$$

$$1 + x\,(a-1) + x\,\frac{x-1}{2}\,(a-1)^2 + x\,\frac{(x-1)}{2}\;\frac{(x-2)}{3}\;(a-1)^3 + \dots$$

Representing the coefficient of x, which is $(a-1) - \frac{1}{2}(a-1)^2 + \frac{1}{3}(a-1)^3 \dots$ by A, those of x^2, x^3 . . . by B, C, &c. the above becomes

$$a^x = 1 + \text{A}x + \text{B}x^2 + \text{C}x^3 + \dots \qquad (1)$$

$$\therefore a^z = 1 + \text{A}z + \text{B}z^2 + \text{C}z^3 + \dots$$

$$\therefore a^{x+z} = (1 + \text{A}x + \text{B}x^2 \dots)\,(1 + \text{A}z + \text{B}z^2 \dots) = \text{by (1)}\ 1 + \text{A}\,(x+z) + \text{B}\,(x+z)^2 + \dots$$

Equating the coefficients of xz, x^2z, x^3z, &c.

$$2\text{B} = \text{A}\,\text{A} \quad \therefore \quad \text{B} = \frac{\text{A}^2}{2}$$

$$3\text{C} = \text{B}\,\text{B} \quad \therefore \quad \text{C} = \frac{\text{A}^3}{1\cdot2\cdot3}$$

&c.

But by Alg. Art. 222, A = Nap. logarithm of a. Denoting a Napierian log. by l (1) becomes

$$a^x = 1 + la\;x + \frac{(la)^2}{1\cdot2}\,x^2 + \frac{(la)^3}{1\cdot2.3}\,x^3 + \dots \qquad (2)$$

Making $a = \varepsilon$ the base of the Napierian system this becomes

$$\varepsilon^x = 1 + x + \frac{x^2}{1\cdot2} + \frac{x^3}{1\cdot2\cdot3} + \dots \qquad (3)$$

From the above may be deduced remarkable expressions for the sine, cosine, and tangent of an arc.

From (3)

$$\varepsilon^{a\sqrt{-1}} = 1 + a\sqrt{-1} - \frac{a^2}{1\cdot2} - \frac{a^3}{1\cdot2\cdot3}\sqrt{-1} + \frac{a^4}{1\cdot2\cdot3\cdot4}$$

$$\varepsilon^{-a\sqrt{-1}} = 1 - a\sqrt{-1} - \frac{a^2}{1\cdot2} + \frac{a^3}{1\cdot2\cdot3}\sqrt{-1} + \frac{a^4}{1\cdot2\cdot3\cdot4}$$

$$\therefore \varepsilon^{a\sqrt{-1}} + \varepsilon^{-a\sqrt{-1}} = 2\left(1 - \frac{a^2}{1\cdot2} + \frac{a^4}{1\cdot2\cdot3\cdot4} - \dots\right) = 2\cos a$$

(App. I. Art. 8.)

$$\varepsilon^{a\sqrt{-1}} - \varepsilon^{-a\sqrt{-1}} = 2\sqrt{-1}\,\left(a - \frac{a^3}{1\cdot2\cdot3} + \dots\right) = 2\sqrt{-1}\sin a$$

$$\therefore \cos a = \tfrac{1}{2}\left(\varepsilon^{a\sqrt{-1}} + \varepsilon^{-a\sqrt{-1}}\right) \qquad (4)$$

$$\sin a = \frac{1}{2\sqrt{-1}}\left(\varepsilon^{a\sqrt{-1}} - \varepsilon^{-a\sqrt{-1}}\right) \qquad (5)$$

Dividing (5) and (4) and multiplying numerator and denominator by $\varepsilon^{a\sqrt{-1}}$ we have

$$\tan a = \frac{1}{\sqrt{-1}}\;\frac{\varepsilon^{2a\sqrt{-1}} - 1}{\varepsilon^{2a\sqrt{-1}} + 1}$$

17. Squaring the expression

$$\cos a + \sin a\sqrt{-1} \qquad (1)$$

there results

$$\cos^2 a - \sin^2 a + 2\cos a \sin a\sqrt{-1} = \cos 2a + \sin 2a\sqrt{-1} \quad \left\{\text{Art. 71, (1) (2)}\right\}$$

Multiplying by (1) we get, applying the forms for sin and cos of $(a + b)$

$$(\cos a + \sin a\sqrt{-1})^3 = \cos 3a + \sin 3a\sqrt{-1}$$

Finally, proceeding in the same way would be obtained

$$(\cos a + \sin a\sqrt{-1})^n = \cos na + \sin na\sqrt{-1} \qquad (2)$$

Making a negative (2) becomes

$$(\cos a - \sin a\sqrt{-1})^n = \cos na - \sin na\sqrt{-1} \qquad (3)$$

18. *To find the n^{th} power of the sine and cosine.*

Assume

$$\begin{aligned} \cos a + \sin a\sqrt{-1} &= z \\ \cos a - \sin a\sqrt{-1} &= u \end{aligned} \qquad (1)$$

By addition,

$$2\cos a = z + u$$

$$\therefore 2^n \cos^n a = z^n + nz^{n-1}u + \frac{n(n-1)}{1\cdot 2}z^{n-2}u^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^{n-3}u^3 + \ldots$$

$$= z^n + nz^{n-2}zu + \frac{n(n-1)}{1\cdot 2}z^{n-4}z^2u^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^{n-6}z^3u^3 + \ldots$$

But $z\,u = 1$ $\therefore$ $z^2u^2 = 1$ &c. $\therefore$ these factors will disappear.
Also by the preceding article, form. (2)

$$\begin{aligned} z^n &= \cos na + \sin na\sqrt{-1} \\ z^{n-2} &= \cos(n-2)a + \sin(n-2)a\sqrt{-1} \end{aligned}$$

Making these substitutions

$$2^n \cos^n a = \cos na + n \cos (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \cos (n-4)\, a + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

$$\cos (n-6)\, a + \ldots \qquad (2)$$

$$+ \left\{ \sin na + n \sin (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \sin (n-4)\, a + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \right.$$

$$\left. \sin (n-6)\, a + \ldots \right\} \sqrt{-1}$$

When n is a positive whole number, the second part of the series vanishes, and the first part extends only as far as a term containing the factor $(n-n)$. The reason why the second part vanishes will be seen by considering that the arcs in the terms preceding that which with the following terms contains $n-n$ will be negatives of the first, second, &c. terms; their sines will be equal therefore with contrary signs, and the coefficients are also evidently equal at equal distances from the centre, and if n be even, the arc in the middle term, and consequently its sine, will be zero.

Again, take the difference of equations (1) and multiply it by $-\sqrt{-1}$

$$2 \sin a = (z-u)(-\sqrt{-1})$$

and the binominal formula proceeding as above gives

$$2^n \sin^n a = \left\{ \begin{array}{l} \cos n\, a - n \cos (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \cos (n-4)\, a - \ldots \\ + \left\{ \sin n\, a - n \sin (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \sin (n-4)\, a - \ldots \right\} \sqrt{-1} \end{array} \right\}$$

$$(-\sqrt{-1})^n \qquad (3)$$

If n be even the second part of the series vanishes. If also n be of the form $4m$, $(-\sqrt{-1})^n = 1$. But if $n = 4m + 2$, then $(-\sqrt{-1})^n = -1$, and our formula becomes

$$2^n \sin^n a = \pm \left\{ \cos n\, a - n \cos (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \cos (n-4)\, a - \ldots \right\}$$

The upper sign applying when $n = 4m$, the lower when $n = 4m + 2$

If n be an odd number, the first part of (3) vanishes, and the formula becomes, $\sqrt{-1}$ outside and inside the parenthesis uniting,

$$2^n \sin^n a = \pm \left\{ \sin n\, a - n \sin (n-2)\, a + \frac{n(n-1)}{1 \cdot 2} \sin (n-4)\, a \ldots \right\}$$

the upper sign of which is to be used when $n = 4m + 1$ and the lower sign when $n = 4m + 3$.

For examples of the applications of these forms see page 101.

PROBLEM I.

19. To determine the area of a plane triangle when any three parts except the three angles are given.

1. Let two sides, a, c, and the included angle B be given. (See fig. Art. 67.)

The area of the triangle is expressed by $\frac{1}{2}$ BC.AD; but AD = AB sin B; hence the expression for the area, in terms of the given quantities, is,

$$\text{area} = \tfrac{1}{2}\, a\, c \sin B \qquad (1)$$

2. Let two angles, B, A, and the interjacent side c, be given

Then, since

$$\sin C : \sin A :: c : a$$

we have

$$a = \frac{\sin A}{\sin C}\, c \;\therefore\; ac \sin B = \frac{\sin A \sin B}{\sin C}\, c^2$$

hence the expression for the area is by (1)

$$\text{area} = \tfrac{1}{2}\, \frac{\sin A \sin B}{\sin C}\, c^2$$

3. Let the three sides be given.

By Art. 73, (2)

$$\sin \tfrac{1}{2} B = \sqrt{\frac{(\frac{1}{2} s - a)(\frac{1}{2} s - c)}{ac}}$$

Also, by (4) of the same article

$$\cos \tfrac{1}{2} B = \sqrt{\frac{\frac{1}{2} s (\frac{1}{2} s - b)}{ac}}$$

$$\therefore 2 \sin \tfrac{1}{2} B \cos \tfrac{1}{2} B, \text{ or (Art. 71) } \sin B = \frac{2}{ac} \sqrt{\tfrac{1}{2} s (\tfrac{1}{2} s - b)(\tfrac{1}{2} s - a)(\tfrac{1}{2} s - c)}$$

Consequently, by substituting this value of sin B in (1) we have

$$\text{area} = \sqrt{\tfrac{1}{2} s (\tfrac{1}{2} s - a)(\tfrac{1}{2} s - b)(\tfrac{1}{2} s - c)}$$

which formula furnishes the well known rule, given in all books on mensuration, for the area of a triangle when the three sides are given.

These expressions for the area of a plane triangle are all adapted to logarithmic computation.

20. In the solution of certain astronomical problems involving the case of solution of triangles at Art. 76, the logarithms of a and b are given, but not the sides themselves; we can very easily calculate $\frac{A-B}{2}$ without knowing the sides.

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$= \frac{1 - \frac{b}{a}}{1 + \frac{b}{a}} \cot \frac{C}{2}$$

Assume $\frac{b}{a} = \tan \phi$

$$\tan \frac{A-B}{2} = \frac{1 - \tan \phi}{1 + \tan \phi} \cot \frac{C}{2}$$

$$= \tan\ (45^\circ - \phi)\ \cot \frac{C}{2}$$

$$\therefore \text{log. tan } \frac{A - B}{2} = \text{log. tan } (45^\circ - \phi) + \text{log. cot } \frac{C}{2} - \text{log. R.}$$

The angle ϕ is known from the equation

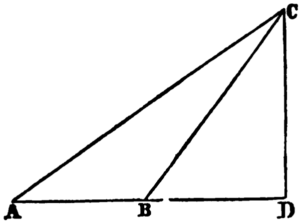
$$\tan \phi = \frac{b}{a}$$

Whence $$\text{log. tan } \phi = \text{log. R} + \text{log. } b - \text{log. } a$$

The angle $\frac{A - B}{2}$ thus becomes known from the logs. of a and b, without calculating a and b. In the same way we may have

$$\cot \frac{A - B}{2} = \tan\ (45^\circ + \phi)\ \tan \frac{C}{2}$$

And $\therefore$ log. cot $\frac{A - B}{2}$ = log. tan. $(45^\circ + \phi)$ + log. tan $\frac{C}{2}$ — log. R.



21. A person on one side of a river observes a building CD on the opposite side, and takes the angle of elevation B = 55° 54′, at the place where he stands, then going back the distance BA = 100 feet, he again takes the angle of elevation A, and finds it to be A = 33° 20′. The height of the building is required.

The problem may be solved as follows.

Taking CD for radius, DB will be the tangent of the angle DCB, and DA, the tangent of DCA, therefore, AB is the difference of those tangents. By the table of natural sines and cosines,*

$$\begin{aligned} \text{nat. tan } 56^\circ\ 40' &= 1{\cdot}520426 \\ \text{nat. tan } 34^\circ\ 6' &= {\cdot}677091 \\ \text{difference} &= {\cdot}843335 \end{aligned}$$

$$\therefore {\cdot}843335 : 1 :: 100 : 118{\cdot}57\ \textit{Ans.}$$

PROBLEM II.

From the top of a mountain three miles high, the angle of depression of a line tangent to the earth's surface is taken, and found to be 2° 13′ 27″; it is required thence to determine the diameter of the earth, supposing it to be a perfect sphere.

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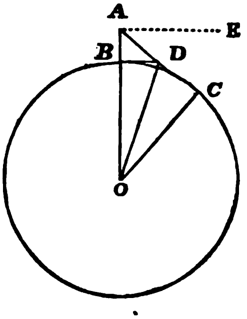
* A table of natural tangents which some collections of tables contain is often convenient.



Let O be the centre of the earth, BA the mountain, AC the visual ray or line touching the earth's surface at C. Draw the tangent BD, and join OD, OC; then the angle of depression EAC being given, we have also the angle BAD, the complement of it, equal to 87° 46′ 33″. Also since the tangents BD, CD, are equal (Geom. p. 83), we have the angle BOD = DOC = ½ comp. A = 1° 6′ 43½″, and, therefore, BDO = 88° 53′ 16½″.

Now in the right-angled triangle ABD we have

BD = AB tan A;

and in the right-angled triangle OBD,

OB = DB tan BDO;

hence, by substitution,

OB = AB tan A tan BDO;

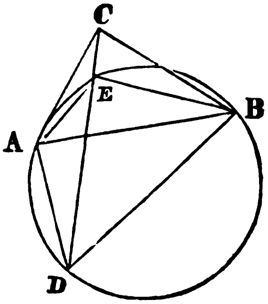
the computation is, therefore, as follows:

A B	3	log.	0·47712
A	87° 46′ 33″	log. tan	11·41074
BDO	88° 53′ 16½″	log. tan	11·71193
OB	3979·15	log.	3·59979

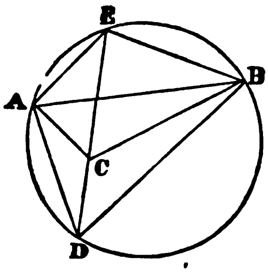
hence the diameter is 7958·3 miles.

PROBLEM III.

Given the distances between three objects, A, B, C, and the angles subtended by these distances at a point D in the same plane with them; to determine the distance of D from each object.



Let a circle be described about the triangle ADB, and join AE, EB, then will the angles ABE, BAE, be respectively equal to the given angles ADE, BDE (Geom. p. 44), thus all the angles of the triangle AEB are known, as also the side AB; we may find, therefore, the remaining sides AE, EB. Again, the sides of the triangle ABC being known, we may find the angle BAC; hence the angle CAE becomes known, so that in the triangle CAE we shall have the two sides AE, AC, and the included angle given, from which we may find the angle AEC in fig. 1, or the angle ACE in fig. 2,



and thence its supplement AED or ACD ; this with the given side AE and angle ADE, in the first figure, or with the given side AC, and angle ADC in the second, will enable us to find AD, one of the required lines, and thence DC and DB the other two.

Or the solution may be conducted more analytically as follows:

Put x for the angle DAC, and x' for the angle DBC ; also call the given angles ADC, BDC α and α' then a, b, c, representing as usual the sides opposite to A, B, C, we have

$$\frac{\sin \alpha}{\sin x} = \frac{b}{\text{DC}} \quad \frac{\sin \alpha'}{\sin x'} = \frac{a}{\text{DC}} \tag{1}$$

$$\therefore \frac{\sin \alpha \sin x'}{\sin \alpha' \sin x} = \frac{b}{a} \therefore a \sin \alpha \sin x' = b \sin \alpha' \sin x \tag{2}$$

This is one equation between the unknown quantities x, x'. Another is easily obtained ; for since the four angles of the quadrilateral ABCD make up four right angles or 360°, we have $x + x' + \alpha + \alpha' + \text{ACD} + \text{BCD} = 360°$; the sum of the two latter angles may become known, since in the triangle ABC the angle C is determinable from the three given sides ; therefore all the terms in the first member of this equation are known except x and x'. Call the sum of the known quantities β, and we shall thus have $x' = \beta - x$, and, consequently by substitution, equation (2) becomes

$$a \sin \alpha \sin (\beta - x) = b \sin \alpha' \sin x$$
$$= a \sin \alpha (\sin \beta \cos x - \cos \beta \sin x) ;$$

or dividing by $\sin x$

$$b \sin \alpha' = a \sin \alpha (\sin \beta \cot x - \cos \beta)$$
$$\therefore \cot x = \frac{b \sin \alpha'}{a \sin \alpha \sin \beta} + \frac{\cos \beta}{\sin \beta}$$
$$= \frac{b \sin \alpha'}{a \sin \alpha \sin \beta} + \cot \beta$$

The first term of this second member may be easily calculated by logarithms, and this added to the natural cotangent of β gives the nat. cot. of x, and thence x' is known from the equation $x' = \beta - x$, and CD from either of the equations (1).

This problem has a useful application in the survey of harbors.

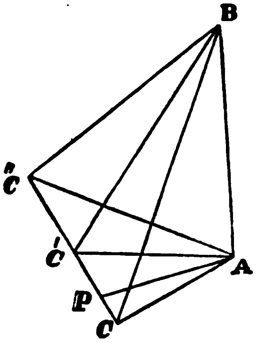
Let the angles be taken with a sextant, from a boat, at a point where a sounding is made, to three stations on the shore. After having drawn upon a map the triangle, of which these three stations are the vertices, the following simple and elegant construction will determine the point where the sounding was made.

Upon the line joining two of the stations, on the map, make a segment, capable of containing the angle observed from the place of sounding, and subtended by this line (Plane Geom., Prob. 21) ; upon a line joining one of these two stations and the third, make another segment that will contain the angle observed to be subtended by this last line, and the intersection of the arcs of these two segments will determine the point on the map, corresponding to that at which the sounding was made.

PROBLEM IV.

Given the angles of elevation of an object taken at three places on the same horizontal straight line, together with the distances between the stations; to find the height of the object and its distance from either station.

Let AB be the object, and C, C′, C″, the three stations, then the triangles BCA, BC′A, BC″A, will all be right angled at A; and, therefore, to radius BA, AC, AC′, AC″, will be the tangents of the angles at B, or the cotangents of the angles of elevation; hence, putting a, a', a'', for the angles of elevation, x for the height of the object, and a, b, for the distances CC′, C′C″, we shall have



$$AC = x \cot a,\ AC' = x \cot a',\ AC'' = x \cot a''$$

Now, if a perpendicular AP be drawn from A to CC″, we shall have (Geom., p. 34,) from the triangle ACC′

$$AC^2 = AC'^2 + C'C^2 - 2\, C'C \,.\, C'P\,;$$

and from the triangle AC′C″

$$AC''^2 = AC'^2 + C''C'^2 + 2\, C''C' \,.\, C'P\,;$$

that is, we shall have the two equations

$$x^2 \cot^2 a = x^2 \cot^2 a' + a^2 - 2a \,.\, C'P$$
$$x^2 \cot^2 a'' = x^2 \cot^2 a' + b^2 + 2b \,.\, C'P$$

in order to eliminate C′P, multiply the first by b, the second by a, and add, and we shall have

$$x^2 (b \cot^2 a + a \cot^2 a'') = (a + b)\, x^2 \cot^2 a' + ab\,(a + b)$$

$$\therefore\ x = \sqrt{\frac{ab\,(a + b)}{b \cot^2 a + a \cot^2 a'' - (a + b) \cot^2 a'}}$$

If the three stations are equidistant, then $a = b$, and the expression becomes

$$x = \frac{a}{\sqrt{\tfrac{1}{2} \cot^2 a + \tfrac{1}{2} \cot^2 a'' - \cot^2 a'}}$$

The height AB being thus determined, the distances of the stations from the object are found by multiplying this height by the cotangents of the angles of elevation.

EXAMPLE.

22 Given the hypothenuse $a = 6512{\cdot}4$ yards, $b = 6510{\cdot}6$, to find c

By Art. 64

$$\begin{array}{lrl}
 & \log.\cos c = & \log.\text{R} + \log. b - \log. a \\
\text{Now} & \log.\text{R} = & 10 \\
 & \log. b = & \underline{3{\cdot}8136210} \\
 & & 13{\cdot}8136210 \\
 & \log. a = & \underline{3{\cdot}8137411} \\
\therefore & \log.\cos c = & 9.9998799 \\
 & c = & 1^\circ\, 20'\, 50''
\end{array}$$

Upon inspecting the tables that are calculated to seven places of decimals only, it will be seen that, when the angles become very small, the cosines differ very little from each other. The same remark applies, of course, to the sines of angles nearly 90°. In cases, therefore, where great accuracy is required, we may commit an important error by calculating a small angle from its cosine, or a large one from its sine. We must consequently endeavor to avoid this, by transforming the expression employed.

In the example before us, c is a small angle which has been calculated from its cosine; we must, therefore, if possible, calculate this angle by means of its sine, or some other trigonometrical function.

Now, by formula (8), Art. 72, we have generally

$$\sin \tfrac{1}{2} c = \sqrt{\frac{1 - \cos c}{2}}$$

In the present case, $\cos c = \frac{b}{a}$ substituting this in the above equation,

$$\sin \tfrac{1}{2} c = \sqrt{\frac{b - a}{2a}}$$

$$\therefore \quad \log.\sin \tfrac{1}{2} c = \tfrac{1}{2} \log.(b - a) - \tfrac{1}{2} \log. 2a + \log. \text{R}.$$

From which we find

$$\frac{c}{2} = 40'\, 24''$$

$$\text{And} \therefore \qquad c = 1^\circ\, 20'\, 48''$$

Instead of 1° 20′ 50″, as obtained by the former process.

Or c might first be calculated from a and b, and then c by means of its sine.

No angle which is nearly 90° ought to be calculated from its tangent, for the tangents of large angles increase with so much rapidity, that the results, derived from the column of proportional parts found in the tables, cannot be depended on as accurate.

23. The following is the demonstration of formulas (3) and (4) of Art. 77. By Art. 69

$$c^2 = a^2 + b^2 - 2ab \cos c = a^2 + b^2 - 2ab\,(2\cos^2 \tfrac{1}{2} c - 1)$$

$$= (a^2 + b^2 + 2ab) - 4ab \cos^2 \tfrac{1}{2} c = (a+b)^2 \left\{ 1 - \left(\frac{2\sqrt{ab}}{a+b} \cos \tfrac{1}{2} c \right)^2 \right\}$$

Making

$$\sin \phi = \frac{2\sqrt{ab}}{a+b} \cos \tfrac{1}{2} c \qquad (1)$$

we have

$$c^2 = (a+b)^2 (1 - \sin^2 \phi) = (a+b)^2 \cos^2 \phi$$

$$\therefore \qquad c = (a+b) \cos \phi \qquad (2)$$

These forms (1) and (2) are more suitable than (1) and (2) of Art. 77, if b be nearly equal to a, because then $\tan \phi$, which, in Art. 77, was equal to $\frac{2\sqrt{ab}}{a-b} \sin \frac{1}{2} c$ is very large, or ϕ is near 90°, unless c is very small, and when such is the case the increase of the tangent is not proportioned to the increase of the arc, so that the ordinary mode of calculating logarithms not exactly found in the tables would be inaccurate.

USE OF SUBSIDIARY ANGLES.

24. Formulas not adapted to logarithmic computation may often be rendered so by the use of subsidiary angles. Specimens have been given in the last Art. and Art. 77. The following is another example.

To adapt

$$\sin a = \cos l \cos d \cos a + \sin d \sin l$$

to logarithmic computation.

It may be written

$$\sin a = \sin d \left(\sin l + \cos l \frac{\cos d \cos a}{\sin d}\right)$$

Put the fractional part $\frac{\cos d \cos a}{\sin d} = \tan \phi$ (1)

Our form thus becomes

$$\sin a = \sin d \left(\sin l + \cos l \frac{\sin \phi}{\cos \phi}\right)$$

$$= \frac{\sin d}{\cos \phi} (\sin l \cos \phi + \cos l \sin \phi)$$

$$= \frac{\sin d}{\cos \phi} \sin (l + \phi)$$

From which

$$\sin (l + \phi) = \frac{\sin a \cos \phi}{\sin d} \qquad (2)$$

ϕ may be computed from (1) by logarithms, and then $l+\phi$ from (2), and from ϕ and $l+\phi$, l becomes known.

25. *To resolve a quadratic equation by the aid of Trigonometry.*

The general form of such an equation is

$$x^2+px=q$$

and the values of x (see Alg. Art. 183) are

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}=-\sqrt{q}\left\{\frac{p}{2\sqrt{q}}\mp\sqrt{\frac{p^2}{4q}+1}\right\}$$

Put $\frac{4q}{p^2}=\tan^2\phi$ (1) $\therefore x=-\sqrt{q}\left\{\frac{1}{\tan\phi}\mp\frac{\sqrt{\tan^2\phi+1}}{\tan\phi}\right\}$

$$=-\sqrt{q}\,\frac{1\mp\sec\phi}{\tan\phi}=-\sqrt{q}\,\frac{\cos\phi\mp 1}{\sin\phi}$$

But

$$\frac{\cos\phi+1}{\sin\phi}=\frac{2\cos^2\frac{1}{2}\phi}{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}=\cot\tfrac{1}{2}\phi$$

and

$$\frac{\cos\phi-1}{\sin\phi}=\frac{-2\sin^2\frac{1}{2}\phi}{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}=-\tan\tfrac{1}{2}\phi$$

The two values of x therefore will be

$$x_1=\frac{-\sqrt{q}}{\tan\frac{1}{2}\phi}\ (2),\ x_2=\sqrt{q}\tan\tfrac{1}{2}\phi \qquad (3)$$

Logarithms may be applied to the formulas (1), (2), and (3).

If p and q be negative the following forms should be used, which may easily be deduced in a similar manner. Put $\frac{4q}{p^2}\sin^2\phi$, then

$$x_1=\tfrac{1}{2}p\,(1+\cos\phi)=p\cos^2\tfrac{1}{2}\phi$$

$$x_2=\tfrac{1}{2}p\,(1-\cos\phi)=p\sin^2\tfrac{1}{2}\phi^*$$

For the resolution of a cubic equation by the aid of Trigonometry, see Alg. Art. 378.

26. *To find the increment of the sine, tangent, &c. corresponding to a small increment of the angle.*

Let a represent the angle, i its increment, and $\delta\sin a$ the corresponding increment of the sine. Then

---

* When the coefficients of a quadratic are large, the trigonometric mode of solution is convenient.

$$
\begin{aligned}
\delta \sin a &= \sin (a+i) - \sin a \\
&= \sin a \cos i + \cos a \sin i - \sin a \\
&= \cos a \sin i - \sin a\,(1 - \cos i) \\
&= \cos a \sin i\,\left(1 - \tan a \frac{2 \sin^2 \frac{1}{2} i}{\sin i}\right) \quad \text{(Art. 72.)} \\
&= \cos a \sin i\,\left(1 - \tan a \frac{(2 \sin^2 \frac{1}{2} i)}{2 \sin \frac{1}{2} i \cos \frac{1}{2} i}\right) \\
&= \cos a \sin i\,(1 - \tan a \tan \tfrac{1}{2} i)
\end{aligned}
$$

The increment i being supposed very small, $\tan \frac{1}{2} i$ will be very small also, and unless $\tan a$ be large, the second term in the parenthesis may be omitted. Then since $\sin i$ is equal to i very nearly, i being very small, it follows that the ratio of $\delta \sin a$ to i is $\cos a$. In other words, the difference of the sines of two angles is proportional to the difference of the angles when the difference is small.

When $\tan a$ is large this principle fails, which is the case with arcs near 90°.

The reasoning for $\delta \cos a$ is very similar.

For the tangent it is as follows:

$$
\begin{aligned}
\delta \tan a = \tan (a+i) - \tan a &= \frac{\sin (a+i)}{\cos (a+i)} - \frac{\sin a}{\cos a} \\
&= \frac{\sin (a+i) \cos a - \cos (a+i) \sin a}{\cos a\,(\cos a \cos i - \sin a \sin i)}
\end{aligned}
$$

The numerator is equal to $\sin \left\{ (a+i) - a \right\} = \sin i$

$$
\therefore\ \delta \tan a = \frac{\sin i}{\cos^2 a \cos i\,(1 - \tan a \tan i)} = \sec^2 a \tan i \cdot \frac{1}{1 - \tan a \tan i}
$$

which if i be very small, and a not near 90°, reduces to $\delta \tan a = \sec^2 a \tan i$.

EXERCISES.

(1). Prove $\operatorname{vers} (180^\circ - a) = 2 \operatorname{vers} \frac{1}{2} (180^\circ + a) \operatorname{vers} \frac{1}{2} (180^\circ - a)$.

(2). Find the numerical values of $\sin 15^\circ$, $\operatorname{vers} 15^\circ$, $\sin 9^\circ$, $\cos 12^\circ$.

(3). Prove $\tan 50^\circ + \cot 50^\circ = 2 \sec 10^\circ$.

(4). If $a + b + c = 90^\circ$, prove

$$\tan a \tan b + \tan a \tan c + \tan b \tan c = 1$$

and

$$\cot a + \cot b + \cot c = \cot a \cot b \cot c$$

and

$$\tan a + \tan b + \tan c = \tan a \tan b \tan c + \sec a \sec b \sec c$$

(5). If $\cos v = \dfrac{\cos u - e}{1 - e \cos u}$ prove $\tan \frac{1}{2} v = \sqrt{\dfrac{1+e}{1-e}} \tan \frac{1}{2} u$*

(6). Prove the radii of the inscribed and circumscribed circles of a regular polygon of any given number (n) of sides to be for the former $r = \frac{1}{2} a \cot \dfrac{180^\circ}{n}$, a being the length of one side of the polygon, and for the latter $R = \frac{1}{2} a \operatorname{cosec} \dfrac{180^\circ}{n}$

---

* This is the formula used in the solution of Kepler's problem in Astronomy.

(7.) Prove the area of a regular circumscribed polygon to be

$$n\, r^2 \tan \frac{180^\circ}{n}$$

That of a regular inscribed polygon to be

$$\frac{n}{2}\, r^2 \sin \frac{360^\circ}{n}$$

(8). Prove the area of a regular polygon of n sides, one of which $= a$ to be

$$\tfrac{1}{4}\, n\, a^2 \cot \frac{180^\circ}{n}$$

(9). Prove the radii of the inscribed and circumscribed circles of a triangle to be

$$r = \sqrt{\frac{(\tfrac{1}{2}s - a)\,(\tfrac{1}{2}s - b)\,(\tfrac{1}{2}s - c)}{\tfrac{1}{2}s}},\ \mathrm{R} = \frac{a\,b\,c}{4\sqrt{\tfrac{1}{2}s\,(\tfrac{1}{2}s - a)\,(\tfrac{1}{2}s - b)\,(\tfrac{1}{2}s - c)}}$$

PART II.

SPHERICAL TRIGONOMETRY

AND

PRACTICAL ASTRONOMY.

77. A spherical triangle is formed by three arcs of great circles.

The planes of these arcs produced, form a trihedral angle, the vertex of which is at the centre of the sphere. (Spherical Geom. Prop. 7.)

The angle which two planes make with each other, is called a diedral angle. This angle is equal to the angle formed by two lines, drawn one in each plane and perpendicular to the common intersection of the two planes at the same point. (Geom. of Planes, Def. 6.)

The angles of a spherical triangle are the angles formed by the planes of the arcs. (Spher. Geom., Def. 7.) These are the diedral angles of the trihedral angle mentioned above.

The sides of the spherical triangle are the arcs of the great circles, by which it is bounded. The arcs subtend the plane angles of the trihedral angle, and consequently measure them. The arcs are given in degrees, and since they contain the same number as the plane angles which they subtend, these plane angles may be employed in a demonstration, instead of the arcs, or sides of the spherical triangle; and for a like reason, the diedral angles of the trihedral angle may be employed instead of the angles of the spherical triangle.

If we suppose the trihedral angle, which has its vertex at the centre of a small sphere, to be produced so as to cut out a triangle upon a larger concentric* sphere, the sides of the triangle upon the larger sphere, containing respectively the same number of degrees as the plane angles of the trihedral angle, will contain the same number of degrees respectively as the sides of the spherical triangle cut out by the trihedral angle on the smaller sphere. So that as the number of degrees in the angles and sides which are given or required, and not their absolute length, is taken into consideration in the solution of spherical triangles, the *size* of the sphere need not be regarded.

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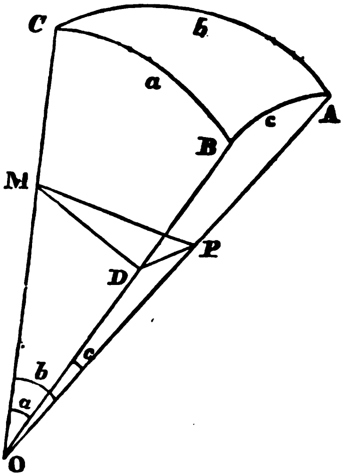
* Having the same centre.

78. Let ABC be a spherical triangle right angled at A. Let O be the centre of the sphere, and let the planes of the arcs which are the sides of the triangle be produced so as to form the trihedral angle whose vertex is at O.

The plane angle COB will contain the same number of degrees as the side *a* of the spherical triangle; the plane angle COA the same number as the side *b*; and AOB the same number as *c*; so that these plane angles may be marked *a*, *b*, and *c*, as in the figure.



It has already been mentioned that the diedral angles of the trihedral angle correspond in the same manner to the angles of the spherical triangle; and that these diedral angles are measured by the angle of two lines, drawn one in each plane, perpendicular to the common intersection of the two planes at the same point. In order to draw these lines so as to be used most conveniently in the following demonstration, take O M = the radius of the tables; draw M P perpendicular to O A, it will be perpendicular to the plane A O B (Geom. of Planes, Prop. 19), since the two planes A O B and A O C are perpendicular to each other, A being by hypothesis a right angle; from P draw P D perpendicular to O B, a line of the plane A O B; join D M; M D will be perpendicular to O B (Geom. of Planes, prop. 8); M D and D P being both perpendicular to O B at the same point D, the angle M D P is the diedral angle of the planes A O B and B O C; or M D P = the angle B of the spherical triangle; O M being equal to radius, M D is the sine of the plane angle *a*, and M P is the sine of the plane angle *b*; in the triangle M D P, right angled at P, we have the proportion (Art. 38)

R : sin D :: M D : M P

substituting for D its equal B, for M D, its value sin a, and for M P, its value sin b, we have

$$R : \sin B :: \sin a : \sin b$$

That is, *the radius is to the sine of either of the oblique angles of a right angled spherical triangle as the sine of the hypothenuse is to the sine of the side opposite that angle.*

79. The solution of astronomical problems forms one of the most useful and agreeable applications of the theory of spherical trigonometry, which branch of mathematics has grown out of the wants of Astronomy.

To illustrate therefore the above and subsequent formulas of spherical trigonometry we shall introduce a few great circles of the celestial sphere. They are so well known that to define them is perhaps superfluous.

The equator is that great circle the plane of which is perpendicular to the axis of the earth. The *axis* being the line about which the earth performs its diurnal rotation. This produced to the celestial sphere becomes the *axis of the heavens* about which all the stars appear to revolve daily.

The ecliptic is a great circle which makes an angle of about 23° 28′ with the equator. It is the path which the sun appears to describe among the stars once a year.

The points in which the two great circles above defined intersect are called equinoctial points.

The one at which the sun crosses the equator in the spring about the 21st of March, is called the vernal equinox.

The other, which is where the sun crosses in the autumn, viz. about the 23d of September, is called the autumnal equinox.

Declination circles are great circles, the planes of which pass through the axis and the circumferences of which all intersect in the *poles* or points where the axis meets the surface of the celestial sphere. They are also called hour circles. The sun appears to move about the earth once in 24 hours; $\frac{360°}{24} = 15°$ is the number of degrees through which the sun moves in an hour.

That declination circle, the plane of which passes through any place on the surface of the earth and the earth's centre, is called the *meridian* of the place.

The angle contained between the meridian of a place and that declination circle which passes through the sun at any given moment, is called the *hour angle* of the sun, and converted into hours, 15° to the hour, will show the time of day, if we reckon from noon instead of midnight as astronomers do.

This time may be either A. M. or P. M. It is what is called apparent time, which varies a little from mean time, the time given by the clocks, in consequence of the slightly unequal motion of the sun in its annual revolution.

The hour angle of a star is similar to that of the sun.

The horizon of any place is a great circle whose plane touches the surface of the earth at that place, and extends to the celestial sphere. This is called the sensible horizon; the real horizon is a plane parallel to this through the centre of the earth. When any of the fixed stars are in question, the distances of which from the earth are so great that its radius is as nothing comparatively, these two horizons may be regarded as coincident. The *zenith* is the pole of the horizon directly overhead. The *nadir* is the opposite pole.

Great circles passing through the zenith and nadir are called *vertical circles.* They are secondaries to the horizon.

The position of a heavenly body is fixed on the celestial sphere, like that of a place on the globe, by its *latitude* and *longitude*, only it must be observed that on the former these are measured from and upon the *ecliptic* instead of the *equator.*

Similar measurements from and upon the celestial *equator* are called the *declination* and the *right ascension*, the former corresponding to the latitude, and the latter to the longitude.*

Longitude upon the earth is reckoned from some fixed meridian, as that of Greenwich.

Longitude upon the celestial sphere is reckoned from the *vernal equinox* which is called the *first of Aries;* right ascension also from the same point; the former upon the ecliptic, the latter upon the equator.

The *azimuth* of a celestial object is an arc of the horizon, comprehended between the meridian of the observer and the vertical circle which passes through the object.

Or it is the angle which these two vertical circles make with each other having its vertex at the zenith.

80. We are now prepared with materials for a practical application of the formulas of spherical trigonometry, and we commence with that already demonstrated.

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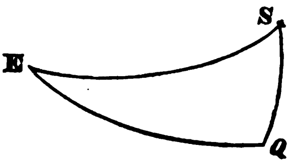
* The symbol for right ascension is AR or R. A.; for declination D, or Dec.



Let E in the annexed diagram be the equinoxial point, EQ a portion of the equator, ES a portion of the ecliptic, S the place of the sun, and SQ a portion of a dec. circle through the sun; then SQ will be the ☉'s declination, which denote by δ, EQ his right ascension, which denote by α, and ES his longitude, which denote by l.

Given the ☉'s declination* equal to 20°, required his longitude.

In the right angled triangle EQS right angled at Q we know E = 23° 28′ the opposite side SQ = 20° required the hypothenuse ES. Hence the proportion

$$R : \sin 23^\circ\ 28' : : \sin l : \sin 20^\circ$$

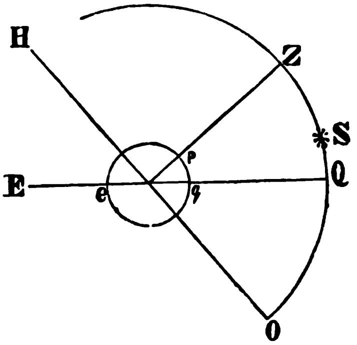
$$\sin l = \frac{R \times \sin 20^\circ}{\sin 23^\circ\ 28'}$$

δ	20°	log. sin 9·53405
E	23° 28′	log. sin 9·60012
l	59° 11′ 26″	log. sin 9·93393

Hence ES = 59° 11′ 26″ the longitude of the sun required.

---

* The declination of the sun may be found rudely by taking its meridian altitude with the same instrument and in the same manner as was described at Art. 11. More accurate instruments and methods will be described hereafter. This observation should be made about noon repeatedly, and the greatest observed altitude will be the meridian altitude. A piece of colored glass will be required for the purpose. Let p be a place on the earth; pq its distance from the equator will be the latitude; this contains the same number of degrees as the arc ZQ between the zenith and celestial equator. Let S be the place of the sun, then SQ will be his declination. Let HO be the horizon, then SO is equal ☉'s meridian altitude, SZ = complement of his altitude, and is called the zenith distance, or coaltitude: SQ = ZQ — SZ or declination = latitude — zenith distance.

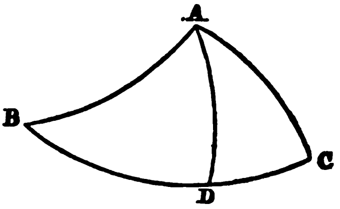


N. B. The altitude of the uppermost point of the circumference of the sun should be first taken, then of the lowermost point, and half their difference added to the latter, or simply half their sum will give the altitude of the ☉'s centre.

Let the student try the following modification of the problem as an exercise.

Given ⊙ = 90° to find his declination.*

81. By means of the proportion for right angled triangles, and of which an application has just been given, one may be derived for triangles in general.



Let ABC be any spherical triangle; let fall from A the arc AD perpendicular to the side BC, the given triangle will be divided into two right angled triangles ABD and ACD. In the right angled triangle ABD we have the proportion (Art. 78)

$$R : \sin B :: \sin AB : \sin AD$$

and in the right angled triangle ACD, the proportion

$$R : \sin C :: \sin AC : \sin AD$$

Multiplying the extremes and means of each of these proportions, we have the equations

$$R \times \sin AD = \sin B \times \sin AB$$

and

$$R \times \sin AD = \sin C \times \sin AC$$

The first members of these equations being the same the second numbers are equal, hence

$$\sin B \times \sin A B = \sin C \times \sin AC$$

substituting for the sides AB and AC the small letters of the same name with the angles opposite to them the last equation may be written

$$\sin B \sin c = \sin C \sin b$$

or

$$\frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

or,

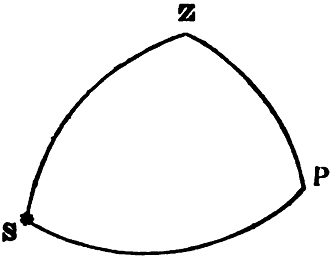
---

* This symbol ⊙ signifies longitude of the sun.

$$\sin B : \sin b :: \sin C : \sin c$$[*]

that is, *the sines of the angles of a spherical triangle are as the sines of the opposite sides.*

EXAMPLE.



Let Z be the zenith, P the pole of the equator, and S the place of a star; ZS will be the zenith distance of the star, ZPS its hour angle, PS its co-declination or polar distance, and SZP its azimuth. Let the azimuth, zenith distance, and hour angle[†] be given, to find the polar distance, which is the complement of the declination; Z, ZS and P are given, and PS required.

SOLUTION.

$$\sin P : \sin Z :: \sin p : \sin z$$

Let $P = 32^\circ\ 26'\ 6''$, $Z = 49^\circ\ 54'\ 38''$, and ZS or $p = 44^\circ\ 13'\ 45''$

$P\ 32^\circ\ 26'\ 6''$	ar. comp. log. sin	0·27056
$Z\ 49^\circ\ 54'\ 38''$	log. sin	9·88369
$p\ 44^\circ\ 13'\ 45''$	log. sin	9·84357
$z\ 84^\circ\ 16'$	log. sin	9·99788

or declination of the star $= 5^\circ\ 44'$

Since the sine of an arc is equal to the sine of its supplement (Art. 15), the required side may be also the supplement of 84° 16′, or 95° 44′ The dec. would then be 5° 44′ south of the equator.

To illustrate this double solution by the diagram, let the student make or conceive to be made the following construction. Draw an arc from S making with PZ an angle equal to Z, meeting PZ in a point which we will call Z′. SZ′ will then be equal to SZ; prolong PZ and PS till they meet in the opposite pole, which we will call P′; a triangle will be formed Z′P′S, in which the angles Z′ and P′, and the side SZ′ will be equal to those given in the above example, but in which the side P′S is the supplement of PS.*

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[*] The student will recollect that a proportion is an equality of ratios, and that ratio, as commonly understood, is the quotient of two quantities. The above is familiarly called the sine proportion.

[†] The zenith dist. and azimuth may be observed with a theodolite or altitude and azimuth instrument, to be described hereafter; the hour angle by a sidereal clock.

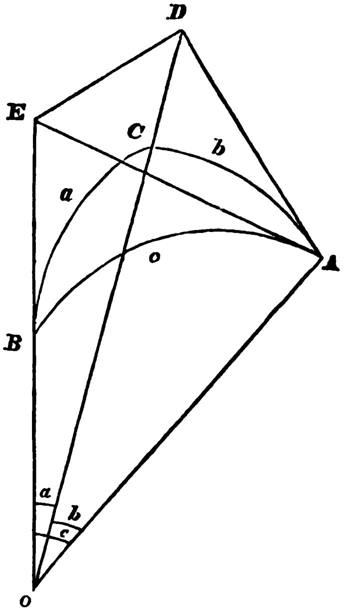
The polar distance of the fixed stars will be found to be always the same, hence they describe circles about the poles in their apparent daily motion.

EXERCISE.

Given a 42° 32′ 19″, A 48° 12′, b 55° 7′ 32″ to find B.

Ans. B 64° 46′ 14″.

82. We shall next demonstrate a formula which will express one of the angles of a spherical triangle in terms of the three sides.



Let ABC be any spherical triangle, O the centre of the sphere; join OA, OB, OC; a trihedral angle is formed having its vertex at O. The plane angles of this trihedral may be called by the same letters as the sides of the spherical triangle for the reason given in Art. 77. By referring to the note of Art. 38, it will be seen that we may choose at pleasure the length of a radius, and the trigonometrical lines will have the same relation as those corresponding to the radius of the tables or any other radius.

Let us take OA as radius, and draw the perpendiculars AD and AE at its extremity and in the planes AOC and AOB; produce these perpendiculars till they meet the lines OC and OB in D and E. AD will be the tangent and OD the secant of the side b of the spherical triangle, and AE the tangent, and OE the secant of the side c.

This being premised, let us take the value of DE, in terms of the other sides and one angle, of each of the two plane triangles DAE and DOE, to both of which it belongs. This may be done by means of the formula

---

* A rule for determining when there are two solutions, and when but one in such cases, is given at p. 196.

$\cos A = R\,\dfrac{b^2 + c^2 - a^2}{2\,bc}$ (Art. 69), from which, taking the value of the square of the side opposite the angle in the formula, we have

$$a^2 = b^2 + c^2 - \frac{2\,b\,c \cos A}{R}$$

in the triangle EAD this formula becomes

$$ED^2 = AD^2 + AE^2 - \frac{2\,AD \times AE \cos A}{R}$$

and in the triangle EOD

$$EP^2 = DO^2 + EO^2 - \frac{2\,DO \times EO \cos a}{R}$$

subtracting the former from the latter of these two equations and observing that in the right angled triangles OAD and OAE

$$DO^2 - AD^2 = OA^2 \text{ and } EO^2 - AE^2 = OA^2$$

there results

$$0 = 2\,OA^2 - \frac{2\,DO \times EO \cos a}{R} + \frac{2\,AD \times AE \cos A}{R}$$

Finding the value of cos A from this equation, we have

$$\cos A = \frac{DO \times EO \cos a - R \times OA^2}{AD \times AE}$$

substituting for OA its value R for DO its value $\sec b = \dfrac{R^2}{\cos b}$ (Art. 33), for EO, $\sec c = \dfrac{R^2}{\cos c}$ for AD its value $\tan b = \dfrac{R \sin b}{\cos b}$ (Art. 32), and for AE its value $\tan c = \dfrac{R \sin c}{\cos c}$ the above expression becomes

$$\cos A = \frac{\dfrac{R^2}{\cos b} \times \dfrac{R^2}{\cos c} \cos a - R^2}{\dfrac{R \sin b}{\cos b}\ \dfrac{R \sin c}{\cos c}} = \frac{R^4 \cos a - R^2 \cos b \cos c}{R^2 \sin b \sin c}$$

or striking out R^2

$$\cos A = \frac{R^2 \cos a - R \cos b \cos c}{\sin b \sin c}$$

or if $R = 1$

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

But the angle A of the plane triangle D A E is the same as the angle A of the spherical triangle (Spher. Geom., Prop. 4) hence, translating the above formula,

The cosine of either angle of a spherical triangle is equal to radius square into the cosine of the side opposite, minus radius into the rectangle of the cosines of the adjacent sides, divided by the rectangle of the sines of the adjacent sides.*

The above formula will serve to calculate one of the angles of a spherical triangle when the three sides are given, if we employ the table of natural sines and cosines; but is unsuitable for the application of logarithms, in consequence of the sign — in the numerator requiring a subtraction to be performed, which operation is impracticable by means of logarithms. We shall therefore derive from this another formula, involving only multiplications, divisions, &c., of the trigonometrical lines contained in it, to which operations logarithms apply.

83. At Art. 74 were derived formulas (2) and (4) for the sum of the sines and difference of the sines of two arcs. In a similar manner two others may be derived for the sum and difference of the cosines. See (7) and (8) of Art. 12, App. I. These four forms would be as follows:

$$\sin m + \sin n = 2 \sin \tfrac{1}{2}(m+n) \cos \tfrac{1}{2}(m-n) \qquad (1)$$
$$\sin m - \sin n = 2 \cos \tfrac{1}{2}(m+n) \sin \tfrac{1}{2}(m-n) \qquad (2)$$
$$\cos m + \cos n = 2 \cos \tfrac{1}{2}(m+n) \cos \tfrac{1}{2}(m-n) \qquad (3)$$
$$\cos n - \cos m = 2 \sin \tfrac{1}{2}(m+n) \sin \tfrac{1}{2}(m-n) \qquad (4)$$

The last is read, *the difference of the cosines of any two arcs is equal to twice the sine of half their sum into the sine of half their difference.*

84. A formula for the cosine of half an angle of a spherical triangle, in terms of the three sides, may now be derived. Resume from Art. 82,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

Add 1 to both members it becomes by (6) of Art. 70,

$$1 + \cos A = \frac{\cos a - \cos (b + c)}{\sin b \sin c}$$

But $\quad 1 + \cos A = 2 \cos^2 \tfrac{1}{2} A$†

---

* We say *either* angle, because in the above demonstration no particular angle was selected.

† Deduced similarly to (8) of Art. 72, by adding (5) and (6) of that article.

∴ (4) Art. 83,

$$\cos^2 \tfrac{1}{2} A = \frac{\sin \frac{1}{2}(b+c+a) \sin \frac{1}{2}(b+c-a)}{\sin b \sin c}$$

or,

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} s \sin \frac{1}{2}(s-a)}{\sin b \sin c}} \qquad (1)$$

i. e. The cosine of half an angle of a spherical triangle is equal to the square root of the sine of half the sum of the three sides into the sine of half the sum minus the side opposite, divided by the rectangle of the sines of the adjacent sides.

In a similar manner may be derived a formula for the sine of half an angle in terms of the three sides.

$$\sin \tfrac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin b \sin c}} \qquad (2)$$

Dividing (2) by (1) there results

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin \frac{1}{2} s \sin (\frac{1}{2} s - a)}} \qquad (3)$$

Since $\sin A = 2 \sin \frac{1}{2} A \cos \frac{1}{2} A$, by multiplying (1) and (2) a formula for sin A may be obtained analogous to (2) of Art. 73.

Any of the above forms (1), (2), (3) may be employed for the solution of a spherical triangle when the three sides are given.

(2) is read thus: *the sine of half either angle of a spherical triangle is equal to radius* into the square root of the sine of half the sum of three sides of the triangle minus one of the adjacent sides, into the sine of half the sum minus the other adjacent side, divided by the rectangle of the sines of the adjacent sides.*

---

* Radius must be introduced as a factor for homogeneity.

The blank form of (2) for examples will be the following.

a		ar. co. log. sin	ar. co. log. sin
b	ar. co. log. sin		ar. co. log. sin
c	ar. co. log. sin	ar. co. log. sin	
s			
$\frac{1}{2}s$			
$\frac{1}{2}s-a$		log. sin	log. sin
$\frac{1}{2}s-b$	log. sin		log. sin
$\frac{1}{2}s-c$	log. sin	log. sin	
	2)	2)	2)
$\frac{1}{2}A$	log. sin		
$\frac{1}{2}B$		log. sin	
$\frac{1}{2}C$			log. sin

EXAMPLE I.—Given $a=120^\circ\ 17'$, $b=75^\circ\ 3'$, $c=48^\circ\ 56'$.

a	$120^\circ\ 17'$			ar. co. log. sine	0·06372	ar. co. log. sine	0·06372
b	$75^\circ\ 3'$	ar. co. log. sine	0·01495			ar. co. log. sine	0·01495
c	$48^\circ\ 56'$	ar. co. log. sine	0·12266	ar. co. log. sine	0·12266		
s	$244^\circ\ 16'$						
$\frac{1}{2}s$	$122^\circ\ 8'$						
$\frac{1}{2}s-a$	$1^\circ\ 51'$			log. sine	8·50897	log. sine	8·50897
$\frac{1}{2}s-b$	$47^\circ\ 5'$	log. sine	9·86472			log. sine	9·86472
$\frac{1}{2}s-c$	$73^\circ\ 12'$	log. sine	9·98106	log. sine	9·98106		
			2)19·98339		2)18·67641		2)18·45236
$\frac{1}{2}A=$	$78^\circ\ 49'\ 40''$	log. sine	9·99169				
$\frac{1}{2}B=$	$12^\circ\ 35'\ 2''\cdot1$			log. sine	9·33820		
$\frac{1}{2}C=$	$9^\circ\ 41'\ 28''\cdot4$					log. sine	9·22618

Ans. $A=157^\circ\ 39'\ 20''$, $B=25^\circ\ 10'\ 4''\cdot2$, $C=19^\circ\ 22'\ 56''\cdot8$

The result is not ambiguous in the application of this formula, for half the angle must be less than 90°, since the whole angle cannot exceed 180°. (Spher. Geom., Prop. 8, Note.)

EXAMPLE II. Given $a=81^\circ\ 17'$, $b=114^\circ\ 3'$, $c=59^\circ\ 12'$.

$$Ans.\begin{cases} A= 62^\circ\ 39'\ 43''\cdot4 \\ B=124^\circ\ 50'\ 50'' \\ C= 50^\circ\ 31'\ 42''\cdot6 \end{cases}$$

This formula, for the solution of a spherical triangle when the three sides are given, is very convenient for calculation by logarithms. Applied to each of the angles separately, it will serve to determine them all.

An interesting astronomical application of this case of solution is to finding the time of day and error of a watch by an altitude of the sun or other heavenly body. See Nautical Astronomy, p. 287.

85. It is proved (Spher. Geom., Prop. 13), that the three angles of a triangle being given, the triangle is determined. A formula for calculating either of the sides when the three angles are given, may be easily derived from that of the last article, by means of the polar triangles.

It is necessary first to premise that the polar triangles of the whole range of triangles will include all possible triangles: for as each side of a triangle passes through all values from 180° to 0°, the opposite angle of the polar triangle will pass through all values from 0° to 180°. Wherefore whatever can be proved of the polar triangles of all possible triangles may be considered as proved for all triangles.

Resume the equation before (1) of the preceding article.

$$\cos \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2}(a+b+c)\sin \frac{1}{2}(b+c-a)}{\sin b \sin c}}$$

For the parts of the triangle in this formula, substitute their equivalents in the polar triangle. It will be remembered (Spher. Geom., Prop. 6), that each angle of a spherical triangle is the supplement of the side opposite in the polar triangle, and vice versâ; hence if a', b', and c' represent the sides, and A', B', C', the angles of the polar triangle, we have

$$A = 180^\circ - a' \quad a = 180^\circ - A', \ b = 180^\circ - B' \text{ and } c = 180^\circ - C'$$

putting these values of the letters A, a, b, and c in their places in the formula above, it becomes

$$\cos \tfrac{1}{2}(180^\circ - a')$$
$$= \sqrt{\frac{\sin \frac{1}{2}(540^\circ - (A' + B' + C'))\sin \frac{1}{2}(180^\circ - (B' + C' - A'))}{\sin(180^\circ - B')\sin(180^\circ - C')}}$$

But $\frac{1}{2}(180^\circ - a') = (90^\circ - \frac{1}{2}a')$ and $\cos(90^\circ - \frac{1}{2}a') = \sin \frac{1}{2}a'$ (Art. 23); also $\frac{1}{2}(540^\circ - (A' + B' + C) = 270^\circ - \frac{1}{2}(A' + B' + C')$ and $\sin(270^\circ - \frac{1}{2}(A' + B' + C')) = -\cos \frac{1}{2}(A' + B' + C')$; also $\frac{1}{2}(180^\circ - (B' + C' - A') = 90^\circ - \frac{1}{2}(B' + C' - A')$ and $\sin(90^\circ - \frac{1}{2}(B' + C' - A')) = \cos \frac{1}{2}(B' + C' - A')$; also $\sin(180^\circ - B') = \sin B'$ (Art. 15) and $\sin(180^\circ - C') = \sin C'$.

Making these substitutions, the formula becomes

$$\sin \tfrac{1}{2} a' = \sqrt{\frac{-\cos \frac{1}{2}(A' + B + C') \cos \frac{1}{2}(B' + C' - A')}{\sin B \sin C'}} \qquad (1)$$

a formula for the sine of half a side in terms of the three angles of a triangle. We may leave out the accents over the letters, which we have employed only to distinguish the polar from the triangle to which it corresponds, and which are superfluous in a general formula.

This formula will undergo a similar modification to that made in the formula preceding (1) of the last article. Represent $A + B + C$ by S, and the formula becomes

$$\sin \tfrac{1}{2} a = \sqrt{\frac{-\cos \frac{1}{2} S \ \cos (\frac{1}{2} S - A)}{\sin B \sin C}} \qquad (2)$$

or, *the sine of half either side of any spherical triangle is equal to radius* into the square root of minus the cosine of half the sum of the three angles into the cosine of half the sum minus the opposite angle, divided by the rectangle of the sines of the adjacent angles.*

A form for the cosine of half a side may be derived in the same manner from the form from which (2) of Art. 84 is derived.

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos \frac{1}{2}(A + B - C) \cos \frac{1}{2}(A + C - B)}{\sin B \sin C}} \qquad (3)$$

from which

$$\cos \tfrac{1}{2} a = \sqrt{\frac{\cos \frac{1}{2}(S - B) \cos \frac{1}{2}(S - C)}{\sin B \sin C}}$$

86. We shall next derive two sets of proportions applicable to the solution of a spherical triangle, the first set when two sides and the included angle are given, and the second when two angles and the included side.

It will be found convenient in the longer analytical processes to represent the angles of a spherical triangle opposite the sides a, b, and c, respectively by α, β, γ. Formula (1) of Art. 84, applied to each of the three angles of a triangle will give

---

* For homogeneity. In the above form $-\cos \frac{1}{2} S$ is always positive, because S is always more than 180°. (See Spher. Geom., Prop. 14.)

$$\mathrm{A}\left\{\begin{array}{lll} \cos^2 \frac{1}{2}\alpha = \dfrac{\sin\frac{1}{2}(b+c+a)\sin\frac{1}{2}(b+c-a)}{\sin b \sin c} & & (1) \\[2ex] \cos^2 \frac{1}{2}\beta = \dfrac{\sin\frac{1}{2}(a+c+b)\sin\frac{1}{2}(a+c-b)}{\sin a \sin c} & & (2) \\[2ex] \cos^2 \frac{1}{2}\gamma = \dfrac{\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a+b-c)}{\sin a \sin b} & & (3) \end{array}\right.$$

and the formula from which (2) of the same article is derived.

$$\mathrm{B}\left\{\begin{array}{lll} \sin^2 \ \alpha = \dfrac{\sin\frac{1}{2}(a+c-b)\sin\frac{1}{2}(a+b-c)}{\sin b \sin c} & & (4) \\[2ex] \sin \frac{1}{2}\beta = \dfrac{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+b-c)}{\sin a \sin c} & & (5) \\[2ex] \sin^2 \frac{1}{2}\gamma = \dfrac{\sin\frac{1}{2}(b+c-a)\sin\frac{1}{2}(a+c-b)}{\sin a \sin b} & & (6) \end{array}\right.$$

By multiplying two of these formulas, and dividing by a third will be obtained

$$\left\{\begin{array}{llll} \text{from}\dfrac{(2)\times(3)}{4} & \text{the formula} & \dfrac{\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha} = \dfrac{\sin\frac{1}{2}(a+b+c)}{\sin a} & (7) \\[2ex] \text{“}\ \dfrac{(5)\times(6)}{(4)} & \text{“} & \dfrac{\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin\frac{1}{2}\alpha} = \dfrac{\sin\frac{1}{2}(b+c-a)}{\sin a} & (8) \end{array}\right.$$

$$\left\{\begin{array}{llll} \text{“}\ \dfrac{(2)\times(6)}{(1)} & \text{“} & \dfrac{\cos\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\cos\frac{1}{2}\alpha} = \dfrac{\sin\frac{1}{2}(a+c-b)}{\sin a} & (9) \\[2ex] \text{“}\ \dfrac{(3)\times(5)}{(1)} & \text{“} & \dfrac{\sin\frac{1}{2}\beta\cos\frac{1}{2}\gamma}{\cos\frac{1}{2}\alpha} = \dfrac{\sin\frac{1}{2}(a+b-c)}{\sin a} & (10) \end{array}\right.$$

Subtracting and adding (7) and (8), and also (9) and (10), applying to the first members of the results of these operations the formulas deduced in Art. 70, and to the second members formulas (5), (6), (7), (8), of Art. 12, App. I., and afterwards to these same second members formula (3) Art. 71, we obtain the following forms.

$$\text{I.}\quad \frac{\cos\frac{1}{2}(\beta+\gamma)}{\sin\frac{1}{2}\alpha} = \frac{\cos\frac{1}{2}(b+c)}{\cos\frac{1}{2}a}$$

$$\text{II.}\quad \frac{\cos\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}\alpha} = \frac{\sin\frac{1}{2}(b+c)}{\sin\frac{1}{2}a}$$

$$\text{III.}\quad \frac{\sin\frac{1}{2}(\beta+\gamma)}{\cos\frac{1}{2}\alpha} = \frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}a}$$

$$\text{IV. } \frac{\sin \frac{1}{2}(\beta - \gamma)}{\cos \frac{1}{2} a} = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2} a}$$

These four forms (and four others derived similarly from repeating for each side formulas (1) and (3) of Art. 85), are known as the Theorem of Gauss.* By division of III. by I., and IV. by II., in both sets will be obtained four additional forms known as Napier's Analogies;† viz.,

$$\text{V. } \tan \tfrac{1}{2} a \tan \tfrac{1}{2}(\beta + \gamma) = \frac{\cos \frac{1}{2}(b - c)}{\cos \frac{1}{2}(b + c)}$$

$$\text{VI. } \tan \tfrac{1}{2} a \tan \tfrac{1}{2}(\beta - \gamma) = \frac{\sin \frac{1}{2}(b - c)}{\sin \frac{1}{2}(b + c)}$$

$$\text{VII. } \cot \tfrac{1}{2} a \tan \tfrac{1}{2}(b + c) = \frac{\cos \frac{1}{2}(\beta - \gamma)}{\cos \frac{1}{2}(\beta + \gamma)}$$

$$\text{VIII. } \cot \tfrac{1}{2} a \tan \tfrac{1}{2}(b - c) = \frac{\sin \frac{1}{2}(\beta - \gamma)}{\sin \frac{1}{2}(\beta + \gamma)}$$

These each converted into a proportion or equality of ratios by writing cot $\frac{1}{2}$ α in the denominator for tan $\frac{1}{2}$ α in V. and VI., and *vice versâ*, in VII. and VIII., will be as follows:

$$\text{IX. } \cos \tfrac{1}{2}(b + c) : \cos \tfrac{1}{2}(b - c) :: \cot \tfrac{1}{2} \alpha : \tan \tfrac{1}{2}(\beta + \gamma) \ldots$$

$$\text{X. } \sin \tfrac{1}{2}(b + c) : \sin \tfrac{1}{2}(b - c) :: \cot \tfrac{1}{2} \alpha : \tan \tfrac{1}{2}(\beta - \gamma) \ldots$$

$$\text{XI. } \cos \tfrac{1}{2}(\beta + \gamma) : \cos \tfrac{1}{2}(\beta - \gamma) :: \tan \tfrac{1}{2} a : \tan \tfrac{1}{2}(b + c) \ldots$$

$$\text{XII. } \sin \tfrac{1}{2}(\beta + \gamma) : \sin \tfrac{1}{2}(\beta - \gamma) :: \tan \tfrac{1}{2} a : \tan \tfrac{1}{2}(b - c) \ldots$$

That is, *the cosine of half the sum of two sides of a spherical triangle is to the cosine of half their difference, as the cotangent of half the included angle is to the tangent of half the sum of the other two angles.*

The second may be repeated in a similar manner, changing cosine into sine and tangent of the half sum into tangent of the half difference of the other two angles.

The third may be translated into ordinary language thus:

The sine of half the sum of two angles of a spherical triangle is to the sine of half their difference as the tangent of half the interjacent side is to the tangent of half the difference of the other two sides.

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[*] Or Gauss equations.

[†] Analogy is a term synonymous with proportion. The first term bears the same analogy or proportion to the second that the third does to the fourth.

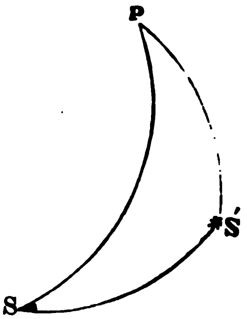
The fourth may be repeated in a similar manner.

These proportions were first given by Lord Napier, who is celebrated for many useful inventions of a similar character, but chiefly for that of logarithms.

We shall now apply the first set to an

EXAMPLE.

The latitudes and longitudes of two places on the earth's surface being given to find the angles which the arc of a great circle joining them makes with their meridians, and their distance apart, the earth being supposed an exact sphere.



Let P be the pole, S and S′ the places, then PS and PS′ will be their colatitudes, and the angle P will be the difference of their longitudes, since P will be measured by an arc at a quadrant's distance on the equator. (Spher. Geom., Prop. 4.) Let the latitudes of the two places be 51° 30′ and 20°; and let their difference of longitude be 31° 34′ 26″. Their colatitudes will be 38° 30′ and 70°. Then we shall know in the above triangle the two sides opposite S and S′ which we will call s and s', and the included angle P. The greater side $s' = 70^\circ$, $s = 38^\circ\ 30'$ and $P = 31^\circ\ 34'\ 26''$. Applying forms IX. and X. of Napier's analogies with the use of logarithms, the half sum and half difference of the unknown angles will be obtained, by the addition and subtraction of which the angles themselves may be found. The remaining side p of the triangle may be found by the sine proportion, or to avoid ambiguity, by form XI.*

The whole computation is contained in the following table.

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* For a highly useful practical application of this problem see Great Circle sailing, App. III.

Arguments.	Logarithms for IX.	Logarithms for X.	Logarithms for XI.
s 38° 30′			
s' 70°			
$\frac{1}{2}(s'+s)$ 54° 15′*	ar. co. log. cos 0·23340	ar. co. log. sin 0·09067	log. tan 10·14273†
$\frac{1}{2}(s'-s)$ 15° 45′	log. cos 9·98338	log. sin 9·43367	
P 31° 34′ 26″			
½ P 15° 47′ 13″	log. cot 10·54863	log. cot 10·54863	
½ (s′ + s) 80° 15′ 40″	log. tan 10·76541		log. cos 9·22828
½ (s′ — s) 49° 47′ 28″		log. tan 10·07297	ar. co. log. cos 0·19005
s′ 130° 3′ 8″			
s 30° 28′ 12″			
½ p 19° 59′ 58″			log. tan 9·56106
p 39° 59′ 56″			

When the first two columns of the above computation are finished, we have the values of ½ (s′ + s) and ½ (s′ — s) the sum of which is equal to s′ and their difference s.

$$\tfrac{1}{2}(s'+s)+\tfrac{1}{2}(s'-s)=130^\circ\ 3'\ 8''=s'$$

$$\tfrac{1}{2}(s'+s)-\tfrac{1}{2}(s'-s)=30^\circ\ 28'\ 12''=s$$

Since we know now all the parts of the triangle except the side p opposite the angle P, that might be found by the proportion (Art. 81), *the sines of the angles are as the sines of the opposite sides.* But to avoid an ambiguity in the result similar to that of Art. 81, and the trouble of determining which of the two results corresponds to the other parts of the triangle now fixed, it is better to employ XI. of Napier's analogies, inverting it as seen in the last column above. This gives $\frac{1}{2}\,p =$ about 20° and $p =$ about $40^\circ = 240$ geographical miles, the distance required.

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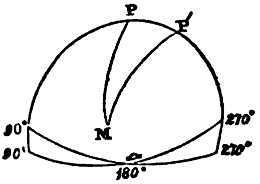
* Had ½ $(s'+s)$ been greater than 90°, its cosine must have been negative, and the first term of the proportion being negative, the fourth must have been negative also, and ½ (s′ + s) would have been the supplement of the angle found in the tables, since the tangent of the supplement is equal to minus the tangent of an arc. (Art. 36.) It is well always to put a small n against those logarithms belonging to trigonometric functions which are negative; then the sign of the result will be indicated by the rule that an even number of negative factors produce a positive result, and an uneven number a negative result.

† This log. need not be found from the tables, but may be obtained by subtracting the ar. comp. of sin ½ $(s'+s)$ above from ar. comp. cos ½ $(s'+s)$ after adding 10 to the latter; for since $\tan=\frac{R\sin}{\cos}$, log. tan $=10+$ log. sin — log. cos. Calling c the ar. comp. log. cos, and s ar. comp. log. sin, this becomes $10+(10-s)-(10-c)=10+c-s$. Q. E. D.

EXAMPLE II.

Given the moon's R.A. $11^h\ 38^m\ 27^s{\cdot}15$. Dec. $4^\circ\ 5'\ 40''{\cdot}4$ N.

To find her latitude and longitude, the obliquity of the ecliptic being $23^\circ\ 27'\ 23''{\cdot}13$.



Let P be the pole of the equator, P′ that of the ecliptic, M the place of the moon; then the angle

$$MPP' = 270^\circ - \text{☽'s R.A.}$$

the side

$$PP' = 23^\circ\ 27'\ 23''{\cdot}13$$

the side

$$PM = 90^\circ - 4^\circ\ 5'\ 40''{\cdot}4$$

To find

$$PP'M = 270^\circ - \text{long. of } M$$

and

$$P'M = 90^\circ \mp \text{☽'s latitude}^{*}$$

Ans. ☽'s lat. $= 1^\circ\ 37'\ 3''{\cdot}3$ N.

long. $= 173^\circ\ 25'\ 57''{\cdot}7$[†]

87. When, in the case considered in Art. 86, the only part required happens to be the side opposite the given angle, the finding of the other two angles then becomes merely a subsidiary operation, and the determination of the required side, by Napier's

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[*] The summer solstice is 90° from the vernal equinox, and is N of the equator. The pole of the ecliptic P′ therefore will be south of P and between it and the 270° point. The right ascension of the moon being nearly 12 hours is nearly 180°, which fixes its place in the diagram.

[†] The determination of the latitude and longitude of a heavenly body from its right ascension and declination as above is one of the most useful problems in Astronomy, the right ascension and declination being observed directly with the astronomical instruments as will be explained in a subsequent part of the work, and the latitude and longitude being required for computing the elements of the orbits of the heavenly bodies.

analogies, seems unnecessarily long. A shorter method of solution is deducible from the fundamental formula, obtained at Art. 82, or

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad (1)$$

For substituting $\cos a \tan a$ for its equal $\sin a$ it becomes

$$\cos c = \cos a\,(\cos b + \tan a \sin b \cos C)$$

Assume

$$\tan a \cos C = \cot \omega = \frac{\cos \omega}{\sin \omega};$$

then

$$\cos c = \cos a \,\frac{\sin \omega \cos b + \sin b \cos \omega}{\sin \omega}$$

$$= \frac{\cos a \sin (\omega + b)}{\sin \omega}.$$

Hence, to find the side c, we must determine a subsidiary angle ω from the equation

$$\cot \omega = \tan a \cos C \qquad (2)$$

after which c is found by the equation

$$\cos c = \frac{\cos a \sin (\omega + b)}{\sin \omega} \qquad (3)$$

EXAMPLE.

1. In a spherical triangle are given $a = 38^\circ\ 30'$ $b = 70^\circ$, and $C = 31^\circ\ 34'\ 28''$, to find c.

a	38° 30′ 0″	log. tan	9.90061	log. cos	9·89354
C	31 34 28	log. cos	9·93042		
ω	55 52 30·5	log. cot	9·83103	ar. co. log. sin	0·08207
b	70°				
$\omega + b$	125° 52′ 30″·5			log. sin	9·90864
c	40°			log cos	9·88425

88. If when two angles and the included side are given, the angle opposite to the given side be the only part required, a similar formula should be employed, deduced as follows. From the fundamental formula (1) above, may be obtained by aid of the polar triangles, the formula

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

which becomes when $\cos A \tan A$ is substituted for $\sin A$,

$$\cos C = \cos A\,(\tan A \sin B \cos c - \cos B):$$

or assuming

$$\tan A \cos c = \cot \omega = \frac{\cos \omega}{\sin \omega}$$

$$\cos C = \cos A \,\frac{\sin B \cos \omega - \sin \omega \cos B}{\sin \omega}$$

$$= \frac{\cos A \sin (B - \omega)}{\sin \omega}$$

Hence, having found a subsidiary angle ω, by the equation

$$\cot \omega = \tan A \cos c \qquad (1)$$

the required angle is determined by the equation

$$\cos C = \frac{\cos A \sin (B - \omega)}{\sin \omega} \qquad (2)$$

89. The formulas for the solution of spherical triangles in general, which have now been demonstrated, apply of course to right angled triangles; but if it be recollected that the trigonometrical lines of the right angle or 90° are either R, 0, or ∞, it will be evident that these formulas may, when thus applied, be much simplified.

The student can easily make the substitutions necessary to change the foregoing formulas into such as apply exclusively to right angled triangles, for himself. We shall not occupy space with them here, but be content with observing that after they have been made, all the formulas which result will be found capable of being expressed in two short rules, or these indeed may be united into a single one.*

Amongst all the convenient and useful inventions of mathematicians, none is more ingenious and beautiful than this, the author of which is the celebrated Lord Napier, whose name we already have had occasion repeatedly to mention in connection with the most happy discoveries for facilitating mathematical operations. The rules are known as

NAPIER'S RULES FOR THE CIRCULAR PARTS.

The circular parts of a right angled spherical triangle are
The two sides including the right angle, called

1. The base.
2. The perpendicular.

And

3. The complement of the hypothenuse
4. The complement of the angle at the base.
5. The complement of the angle at the vertex.

The right angle being entirely left out of consideration in the solution of triangles of this kind, the angle at the base is that included between the base and the hypothenuse ; and the angle at the vertex is that included between the hypothenuse and perpendicular.

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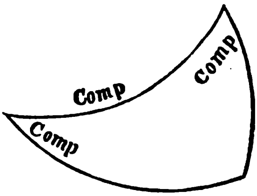
* The mode of deducing them is given in App. II. p. 194.

The circular parts are then the elements of the triangle itself except the right angle, only that the complements of the hypothenuse and oblique angles are the circular parts, instead of these themselves.

The annexed diagram shows of which elements of the triangle the complements are used.



There being five of these circular parts, it is evident that any three of them which you choose to select will either be contiguous or else two will be contiguous, and one will be separated from them by a part on each side.

In the first case, the part intermediate between the other two is called the middle part, and they are called its adjacent parts. In the second case, the part which is separated from the other two is called the middle part, and they its opposite parts. By means of this arrangement, all the relations of a right angled triangle may be expressed in the two following rules of Napier:

1. Radius multiplied by the sine of the middle part is equal to the rectangle of the tangents of the adjacent parts.
2. Radius multiplied by the sine of the middle part is equal to the rectangle of the cosines of the opposite parts.

Or both rules may be given thus: *radius into the sine of the middle part = the rectangle of the tangents of the adjacent parts = the rectangle of the cosines of the opposite parts.*[*] .

The memory will be aided by observing that the words *tangents* and *adjacent* in the second clause of the above rule both contain the letter *a ;* and that the words *cosines* and *opposite* in the last clause both contain the letter *o*.

As the right angle of a right angled spherical triangle is always known, any other two parts being given, the rest may be found by the above rules.

The method of proceeding is as follows: Take the two given parts and one of the required parts, or if but one of the unknown parts be required, take that, you will thus have under consideration three parts of the triangle. One of these three will be middle, and the other two either adjacent or opposite; apply the rule of Napier, and you will have an equation resulting which will contain the two given parts and the required part; make the required part the unknown quantity in the equation, and resolve it, you will thus obtain the value of the required part in terms of the two which were given. By applying logarithms to this value, you will have it in degrees, minutes, and seconds.

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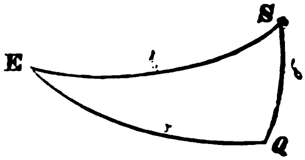
[*] The rule may be read without radius which must be understood as entering the resulting formulas, in accordance with the principles of homogeneity.

EXAMPLE

90. Given the sun's right ascension and declination to find his longitude.



Let the parts of the right angled spherical triangle EQS represent the same circles of the celestial sphere, as at Art. 80; α and δ are given, and l is required.

Of these three parts α, δ and l, α and δ are contiguous, and l is separated from them by a part on each side; therefore l is the middle part and α and δ are opposite parts. Applying Napier's rule, remembering that the complement of the hypothenuse l is to be employed, we have

$$\sin \text{ of comp. } l = \cos \alpha \cos \delta$$

or,

$$\cos l = \cos \alpha \cos \delta$$

91. The same being given, required the obliquity of the ecliptic.

The required part is the angle E in the figure. Of E, α and δ, since the three are contiguous leaving out the right angle, α is the middle part, hence applying the rule of Napier,

$$\sin \alpha = \tan \delta \cot \text{E}$$

We put cot E instead of tan E, because, according to the directions before given, the complements of the oblique angles are to be employed. Taking the value of cot E from the above equation, we have

$$\cot \text{E} = \frac{\sin \alpha}{\tan \delta}$$

The sun's R.A on 1st of May 1850 is $2^h\ 33^m\ 10^s{\cdot}42$, and his declination at the same time $15^\circ\ 3'\ 2''{\cdot}4$ required his longitude and the obliquity of the ecliptic.

a 38° 17′ 36″·30	log. cos 9·89479	log. sin 9·79217
δ 15° 3′ 2″·4	log. cos 9·98484	log. tan 9·42959
l 40° 43′ 5″·86	log. cos 9·87963	
B 23° 27′ 25″·46		log. cot 10·36259

In the solution of the above triangle it will be observed that we have found each of the unknown parts in terms of the two given, and have not employed one of those first calculated to obtain another. This is agreeable to the principle laid down at Art. 41, of plane trigonometry, and the reason is the same. Such a method of proceeding is always practicable in the solution of right angled spherical triangles.

In the examples which we have taken above, we have supposed the base and perpendicular of a right angled spherical triangle given. Any other two parts being given, each of the unknown parts may be calculated by the aid of Napier's rules, in a manner entirely similar to what has been just exhibited.

EXERCISES.

1. In the spherical triangle ABC right angled at A, given the hypothenuse a 65° 5′ and the angle C 48° 12′ to find B, b and c

Ans.	B	64° 46′ 14″
	b	55 7 32
	c	42 32 19

2. Given a 127° 12′, c 141° 11′ to find b, B and C.

Ans.	b	39° 6′ 26″
	B	52 22 24
	C	128 6 26

3. Given the two oblique angles B 111° 11′, C 91° 11′ to find the three sides.

Ans.	a	89° 32′ 28″
	b	111 48 48
	c	91 16 8

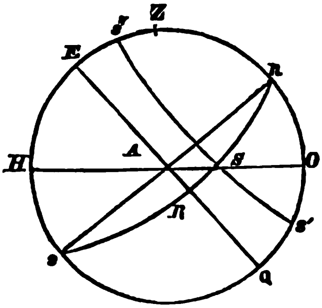
N. B. If the given quantities in a right angled triangle be a side, and its opposite angle, there will be legitimate ambiguity in the solution.

In all other cases no ambiguity properly exists, but to avoid error it is necessary to observe the two following principles.

1. The greater side is opposite to the greater angle.
2. An angle and the opposite side are of the same affection, *i. e.*, both greater or both less than 90°.

EXAMPLE I.

92. Given the sun's declination to find the time of his rising and setting at any place whose latitude is known.



Let *n* E *s* Q represent the meridian of the place, Z being the zenith, and HO the horizon, and let *s*′ *s*″ be the apparent path of the sun on the proposed day, cutting the horizon in S. Then the arc EZ will be the latitude of the place, and consequently EH, or its equal QO, will be the colatitude, and this measures the angle OAQ; also RS will be the sun's declination, and AR, expressed in time, will express the time of sunrise from 6 o'clock, for *n*A*s* is the 6 o'clock hour circle.

Hence, in the right angled triangle SAR, we have given RS, and the opposite angle A to find AR, the time from 6 o'clock.

Required the time of sunrise at latitude 40° 43′, when the sun's declination is 23° 27′

By Napier's rule,

Rad. sin AR = cot A tan RS = tan lat. tan dec.

23° 27′ log. tan 9·63726
40° 43′ log. tan 9·93482
21° 55′ 13″ log. sin 9·57208
4[*]
60)87° 40′ 52″
AR in time 1ʰ 27ᵐ 41ˢ
6
4ʰ 32ᵐ 19ˢ time of sun rising.

SCHOLIUM.

It should be here remarked that the time thus determined is *apparent time*, which is that which would be shown by a clock so adjusted as to pass over 24 hours during one apparent revolution of the sun, or from its leaving the meridian to its return to it again, the index pointing to 12, when the sun is on the meridian. But it is impossible that any clock can be so adjusted, because the interval between the successive returns of the sun to the meridian is continually varying, on account of the unequal motion of the sun in its orbit, and of the obliquity of the ecliptic; each of these varying intervals is called a *true solar day*, and it is the mean of these during the year which is measured by the 24 hours of a well regulated clock, this period of time being a mean solar day; hence, at certain periods of the year, the sun will arrive at the meridian before the clock points to 12, and at other periods the clock will precede the sun; the small interval between the arrival of the index of the clock at 12 and of the sun to the meridian, is called the *equation of time*, and it is given on pages I. and II. of each month of the Nautical Almanac for every day in the month; this correction, therefore, must always be applied to the apparent time determined by trigonometrical calculation to obtain the *mean time* or that shown by a well regulated clock or chronometer, or *vice versâ*, and the Nautical Almanac always indicates whether this correction is additive or subtractive.*

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[*] Degrees are converted into hours by multiplying by 4 and dividing by 60, which is equivalent to dividing by 15.

A third kind of time is called siderial time. A siderial day is the period of revolution of the earth upon its axis with reference to the fixed stars, or it is the time which elapses after a fixed star passes the meridian of any place until the same star comes to that meridian again. Owing to the apparent motion of the sun from west to east along the stars about $\frac{360^\circ}{365\frac{1}{4}}$ daily, occasioned by the real motion of the earth in its annular orbit, the solar day is a little longer than the siderial, because when the meridian of a place has revolved with the earth on its axis from west to east to come under a certain star, the sun which the day before may have been on the meridian with the star having moved a little to the east, the meridian has a little farther to revolve towards the east to come under the sun again, and thus complete the solar day. The difference between the siderial and solar day is about $3^{m}.\ 57^{s}$. The same fixed stars cross the meridian, rise and set about this much later every day. The siderial day is divided into 24 siderial hours, the hour into 60 siderial minutes, and these each into 60 siderial seconds. At pages† 584, 585, 586, 587 of the

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* To solve the above problem very accurately it would be necessary to compute the sun's declination at the time of sunrise as deduced approximately above, and then to go over the calculation again. The Nautical Almanac gives the declination of the sun at noon for every day in the year, and of the process for determining its declination at any other time of the day we shall have numerous examples in Part. V.

† The Nos. of these pages change a little every year.

Greenwich Nautical Almanac are what are called tables of time equivalents. On pages 584, 585 will be found for any given interval of mean solar hours, or minutes, or seconds, the equivalent interval in siderial time. And similarly on pages 586, 587, for any given siderial interval will be found the equivalent mean solar interval.

EXAMPLE.

Required the equivalent of an interval of $7^h\ 28^m\ 30^s$ of mean solar time in siderial time. From page 584 the equivalent

of	7^h	is found to be	$7^h\ 1^m\ 8^s{\cdot}99$
"	28^m	" "	$28^m\ 4{\cdot}59$
"	30^s	" "	$30{\cdot}08$
Ans.		By addition	$7^h\ 29^m\ 43^s{\cdot}66$

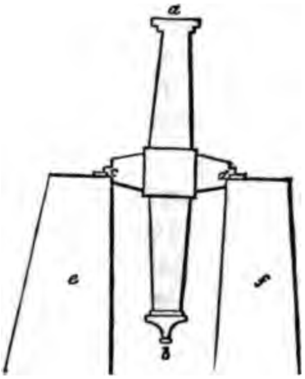
EXAMPLE II.

Required the solar equivalent of $22^{sid.\,h}\ 30^m\ 27^s$ (page 586 N. A.)

22^h	=	$21^h\ 56^m\ 23^s{\cdot}75$
30^m	=	$29\ \ 55{\cdot}08$
27	=	$26{\cdot}93$
Ans.		$22\ \ 26\ \ 45{\cdot}76$

An astronomical clock is one which keeps siderial time. A common clock may be made to do this by shortening a little the pendulum. The weight attached to the pendulum is usually furnished with a screw by which it may be lengthened and shortened at pleasure, and this should be done till the clock goes just 24 hours from the time a star makes its transit over the meridian till the same star makes its meridian transit again.

The exact instant of a star's crossing the meridian is observed with a "Transit Instrument."



This instrument consists of a telescope *a b* supported by a horizontal axis, each half of which *c* and *d* is a hollow cone of brass, at the outer ends of which are short solid cylindrical pivots which rest upon stone pillars *e* and *f*, called piers, the latter being imbedded in a mass of masonry extending a few feet below the surface of the ground, in order that the vibrations occasioned by passing vehicles or the tread of the observers may not be felt. The pivots of the axis do not rest immediately upon the piers, but upon flat pieces of brass about 4 inches square, and an inch in thickness, which are screwed to the top of the stone. These brass pieces have a notch technically called a Y or V from its shape, in which the pivot of the axis rests. The part of the piece of brass having the notch or V is detached and movable, by means of a screw arranged differently at the opposite ends of the horizontal axis *c d*, so that one end may be moved horizontally, and the other vertically.

The exact line of vision directed to a distant object is marked by two threads of spider's web, technically called wires, crossing each other at right angles, at a point in or near the optical axis of the telescope. They are stretched across a ring or diaphragm to which they are fastened with wax, and this ring, which is smaller in diameter than the tube of the telescope, is held in its place by screws passing through the tube, having their heads outside. By loosening the screw on one side of the tube and tightening the other, the diaphragm, and consequently the point in which the wires cross, receives a lateral motion. The diaphragm is placed at the focus of the object glass near the eye end *b*.†

The whole instrument just described has to be so placed that as it turns on the pivots of the horizontal axis the line of vision along the optical axis of the telescope shall describe the plane of the meridian; narrow trap doors in the roof and sides of the transit room serve to expose the meridian to view. As this plane is vertical, if the line of vision above mentioned be exactly perpendicular to the axis *c d*, whilst at the same time the axis is exactly horizontal, and finally the telescope point due north and south, then will the required position be attained. For this, therefore, three adjustments are requisite. 1. The adjustment of the line of collimation* in a perpendicular to the supporting axis *c d*. 2. The adjustment of the supporting axis to a horizontal position. 3. The adjustment of the line of collimation to the meridian.

The method of making these several adjustments we shall describe in their order.

1. *To collimate the instrument.*—Bring the intersection of the wires upon a well defined point of some distant terrestrial object; take the instrument out of the Y's and reverse the supporting axis end for end; bring the telescope upon the same distant point, and if the intersection of the wires covers it exactly, the instrument is collimated; if not, move

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* *i. e.* the line of vision determined by the intersection of the wires.

† For the illumination of the wires see p. 363.

the diaphragm containing the wires by means of the screws at the side of the tube, till the intersection of the wires is brought half way back to cover the distant[*] point; bring the cross wire on the same or some other point again, by means of the screw in the Y, which gives a horizontal motion to the whole instrument, and repeat the process already described; after a few trials the point will be found to be exactly covered by the intersection of the wires in both positions of the telescope. This indicates that the line of collimation, or line determined by the intersection of the cross wires, and the distant point, is exactly perpendicular to the axis on which the instrument turns as the object end of the telescope is elevated or depressed.

2. *To render the supporting axis horizontal.*—This is done by means of a spirit level, of which there are two kinds for the purpose, the hanging level, and the riding or striding level.

The former is suspended by hooks from the pivots of the supporting axis, so as to hang parallel to it underneath. The latter is sustained above the supporting axis by two long feet with notches at their bottoms, by means of which it stands upon the pivots of the axis.

First, to adjust the spirit level itself, place it on the pivots, and by means of the screw in the Y at that extremity of the axis which gives it a vertical motion, bring the long air bubble of the level to reach exactly the same distance on either side of the centre marked with a zero on the level scale above the tube; for which purpose the divisions of this scale are numbered in precisely the same manner on the right and left of the zero. Then reverse the level on the pivots, turning it end for end, and if the bubble still reaches the same distance on both sides of the zero, the level requires no adjustment. If not, make half the correction by filing away the notch in one of the feet, or by means of a screw sometimes added for shortening the foot, and the other half by the screw in the Y. Repeat this process till the adjustment is complete. When the level itself is once

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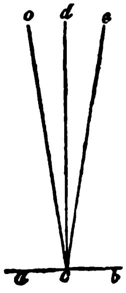
[*] This may be exhibited with the error of collimation exaggerated in the annexed diagram, in which *a b* represents the supporting axis, *c d* the true line of collimation, *c o* the erroneous position of the line of collimation in the first position of the instrument in the direction *c o* of the object, and *c e* the position of the line of collimation in the reversed position.



adjusted, the supporting axis is made horizontal by placing the level upon it, and turning the screw in the Y till the centre of the air bubble is opposite the zero of the scale.

3. *To adjust the instrument to the meridian.*—Observe the instant that some circumpolar* star (the pole star is the best, from the slowness of its motion) crosses the vertical wire of the transit instrument both at its superior and inferior transit, that is above and below the pole. If the interval of time between the superior and inferior transit be the same with that between the latter and the next superior transit of the same star again, the instrument is in the meridian. If not, it is on that side of the meridian on which the arc described by the star between the two transits is shortest, and must be moved a little by means of the screw in the Y, which gives horizontal motion, and the same observations repeated.† When the instrument is once fixed in the meridian, a meridian mark about half a mile distant may be made upon some object, set up if necessary, upon which the vertical wire is to be brought whenever afterwards an observation is to be made.

Method of observing the meridian transit of a star.—Before describing this we shall observe that for diminishing the error of observation there are inserted on each side of the vertical middle wire, one, two, or three others, making three, five, or seven in all. There is also attached to the supporting axis near one of the pivots a graduated circle, the plane of which is perpendicular to that axis, and consequently vertical. This circle is graduated so that the index points to zero when the telescope points to the zenith, or else when the telescope is horizontal, so that when the telescope is directed to a star, the index will mark the zenith distance of the star in the former case, and its altitude in the latter. The star's declination being known from the Nautical Almanac, or from a catalogue, and the latitude of the place of observation being also known, the instrument may be easily set so that when the star makes its meridian transit it will pass through the middle of the field of view of the telescope. For it is only necessary to bear in mind that the declination is the distance of the star from the equator and the latitude is the distance of the zenith from the equator,‡ so that by simple addition or subtraction of these quantities the distance of the star from the zenith is obtained. Thus, if the star be south of the zenith and north of the equator its declination must be subtracted from the latitude to obtain the zenith distance. If the star be south of the equator, or its dec. be S., the dec. must be added to the lat. If the star be N. of the zenith, *i. e.* if its N. dec. exceed the N. lat. of the place, then the lat. must be subtracted from the dec. to obtain the zenith distance. The above definitions for dec. and lat. will always be the best guide. If the instrument be graduated for altitudes instead of zenith distances it is only necessary to recollect that the altitude is the complement of the zenith distance. If an inferior transit, or transit *sub polo* of a circumpolar star is to be taken, it may be convenient to remember that the altitude of the pole is equal to the latitude of the place.*

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* A circumpolar star is one which never sets, but describes daily a circle round the pole of the heavens, the whole of which is visible above the horizon.

† A method of determining the exact deviation from the meridian, and the consequent error in the time of meridian transit, will be presently given.

‡ These measures are all made on the same great circle of the heavens, viz. the meridian of the place of observation, upon which the star is supposed to be at the instant of transit.

The instrument being set to the proper altitude, so that when the star crosses the meridian it will be sure to be seen in the field of view of the telescope, it remains now only to know when to look for its arrival at the meridian, and its consequent appearance in the field. This is shown by the astronomical clock, which is supported upon a stone pier in the same room with the transit instrument. This clock, when correctly set, should indicate the zero of time, or $0^h\ 0^m\ 0^s$ at the exact instant that the vernal equinox is on the meridian; then at the instant any star is on the meridian, the clock would show the distance of that star in time from the vernal equinox or the right ascension of the star.

A minute or two therefore before the clock shows a time equal to the right ascension of the star, (for whose zenith distance or altitude the instrument is set,) as given by the Nautical Almanac or by catalogue, place the eye at the telescope, and the star will be seen entering the field of view, and moving in a direction contrary to its real motion, *i. e.* from west to east† instead of from east to west, because an astronomical telescope inverts. Bring it near the horizontal wire,‡ by means of a clamp and tangent screw attached to the vertical circle, and before the star reaches the first vertical wire in its motion across the field, look at the clock and take up the count of the seconds, which keep by the ear, applying the eye again to the telescope, and note the instant the star crosses or is bisected by the first vertical wire; record the second in a blank book, then look at the clock, and record the minute; the hour may be left till the observation is finished. Take up the count of the second again, and apply the eye to the instrument; by this time the star will be seen approaching the second vertical wire. Observe and record the instant of its transit over that wire in the same way, and so on for all the vertical wires. The sum of the minutes and seconds divided by the number of wires will give the minutes and seconds of the time of passing the middle wire, with a probable error of $\frac{1}{5}$ or $\frac{1}{7}$ (according to the number of wires) of the error if the observation had been made upon the middle wire alone.*

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* For the zenith being 90° from the horizon, and the pole 90° from the equator, the pole will be just as far from the horizon as the zenith is from the equator.

† Unless it be making its transit *sub polo*.

‡ Or between the two horizontal wires if there are two near together, as is sometimes the case.

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* The star will often be seen bisected by a wire between two beats of seconds. The eye then notes how far from the wire the star was at the beat before the bisection and how far at the beat after, and estimates the fraction of a second at which the bisection took place. By the aid of electro-magnetism the exact instant, to a very small fraction of a second, may be not only observed, but recorded without the trouble of keeping count. The arrangement for the purpose is as follows: A wire is made to communicate from one pole of a voltaic battery to the brass work of the clock; the electricity perviates all the brass work, and passes down the pendulum rod; underneath the pendulum a globule of mercury is supported in a little metallic cup from which a wire passes to a magnet, round which it coils, and then passes on to the other pole of the battery. Every time the pendulum vibrates on reaching the lowest point of its arc, it dips into the globule of mercury for an instant, and thus a communication is made between the two poles of the battery, and the magnet acts, drawing back a little hammer which is armed with a sharp point that pricks a narrow strip of paper made to pass along under it, at a uniform rate, by clockwork. The intervals between the points on the paper will correspond to seconds. Two wires communicate also from the opposite poles of a battery (one of them coiling round a magnet close beside the magnet already mentioned) to a wooden block held in the hand of the observer at the instrument, by touching a button in which he connects the wires communicating with it, and the magnet then acts, causing a small hammer to prick with its sharp point the narrow strip of paper a little on one side of the line of points which mark the seconds. The precise position in the interval between two even seconds, of the instant of bisection of the star by the wire, is thus indicated with great precision, and by applying a scale with a vernier, the fraction of a second may be obtained to thousandths, the space on the strip of paper corresponding to a second being usually from half an inch to an inch. The observer at the end of the observation notes on the clock the minute with which the observation closes, and writes it with the hour on the strip of paper in pencil.

The even minutes in the line of dots which marks the seconds on the strip of paper are indicated by the omission of a dot, which is effected as follows. To the axis which carries the second hand of the clock is attached a fork of two prongs, projecting perpendicularly from the axis; when the second hand has made a complete revolution of the clock dial, the two prongs of the fork dip into two globules of mercury communicating by it with the two poles of the battery, one of which wires is the same that communicates the electricity to the brass work of the clock from which

The interval between the wires ought to be exactly the same. As this is scarcely attainable in practice, the mean of the times of transit over all the wires obtained as above will be the time of transit over an imaginary wire situated very near the middle wire. If, by sudden cloudiness or any other accident the transits over some or all the wires but one should be lost, the time of transit over the imaginary middle wire may be obtained as follows:

Having made a complete observation of the times of transit of some star over all the wires, take the difference between the mean of all and the time of transit over each wire. Multiply the intervals thus obtained by the cosine of the star's declination (see Spher. Geom., Prop. 2, Cor. 6, and see Navigation, Art. 99*), and the products will be the equatorial interval between that wire and the imaginary middle wire, or the time that would be occupied by a star situated on the equator in traversing the same interval.† The equatorial intervals being once obtained, to know the time of any star's passing the imaginary middle wire from the time of its passing any other wire, *divide* the equatorial interval between this wire and the middle wire by the cosine of the star's declination.

it goes down the pendulum as before described. When the connexion is made by the fork dipping into the two globules of mercury the electricity goes back to the battery by the shortest path instead of taking the course down the pendulum, and that beat is lost on the magnet.

A number of rapid strokes of the button, which impress a corresponding number of points on the paper, serve to indicate the commencement of the observation.

* The principle alluded to here, which is of frequent use in astronomy, may be stated thus; the arc on a great circle comprehended between two of its secondaries, is to the arc of a small circle parallel to the primary comprehended between the same secondaries, as unity is to the cosine of the distance of the parallel small circle from the primary, this distance being measured on one of the secondaries.

The fourth term of the proportion, instead of the cosine of the distance from the primary, may be the sine of the distance from the pole of the primary or the point in which the two secondaries meet.

† For as the length of arc passed over between two hour circles in the same time on the equator, and on a parallel of declination, is as the cosine of the declination to 1, so the times of passing over the same length of arc (as for instance that included between the wires) on the equator and on a parallel of declination will be in the same ratio.

EXAMPLE.

The transit of the star β Ursæ Minoris, whose declination was $74^\circ\ 46'$ was observed as follows:

Wires.	h.	m.	s.
I.	14	48	11
II.	14	49	16 •2
III.	14	50	22 •5
IV.	14	51	28
V.	14	52	33 •2
5)	70^h	250^m	$110^s{\cdot}9$
	14^h	50^m	$22^s{\cdot}18$ Time imag. mid. wire

Intervals.		
I. and III.*	+	$131^s{\cdot}18$
II. and III.	+	$66^s{\cdot}16$
IV. and III.	—	$65^s{\cdot}82$
V. and III.	—	$131^s{\cdot}02$

I. and III.

$131^s{\cdot}18$ log. 2•11787
$74^\circ\ 46'$ log. cos 9•41954
equat int^s. $34^s{\cdot}47$ log. 1•53741

II. and III.

$66^s{\cdot}16$ log. 1•82060
9•41954
17 •38 log. 1•24014

IV. and III.

65•82 log. 1•81836
9•41954
equat int^s. 17•29 1•23790

V. and III.

131•02 log. 2•11734
9•41954
34•42 log. 1•53688

March 8th, 1850. The star ε, Canis Majoris, was observed on the Vth wire only. Time of transit over that wire $6^h\ 54^m\ 3^s{\cdot}8$

equat. interv. $34^s{\cdot}42$ log. 1•53681
star's dec. (N. Alm.) $28^\circ\ 46'\ 16''{\cdot}89$ log. cos 9•94277
int. on par. of dec. $39^s{\cdot}27$ log. 1•59404
Correction —39•27
Transit imaginary middle wire $6^h\ 53^m\ 24{\cdot}53$

To compute the effect of error of level upon the time of meridian transit.— It will be first necessary to determine the inclination of the supporting axis to the horizon. For this purpose place the striding level on the pivots, and take the readings at both ends; suppose as in the diagram the

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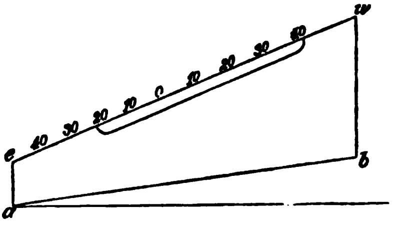
* III. here and below stand for the imaginary middle wire.

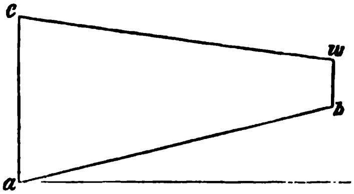


west end reads 40 and the east 20. It is evident that each end of the bubble stood at 30 when the level was horizontal, and that each has moved 10 divisions,* which is obtained by taking the difference between the readings of the east and west end, and dividing by 2. So that if i denote the inclination of the supporting axis ab to a horizontal, and i' the inclination of the level to ab, $i + i'$ will denote the inclination of the level to the horizon, and we have (ω denoting the west reading and e the east).

$$\frac{\omega - e}{2} = i + i' \qquad (1)$$

Reversing the level, supposing i' to be greater than i, the east end will now be the highest, but the inclination to the horizon will no longer be the sum, but the difference of the inclinations i and i', hence

$$\frac{e' - \omega'}{2} = i' - i \qquad (2)$$



subtracting (2) from (1), and dividing by 2 we have

$$\frac{(\omega + \omega') - (e + e')}{4} = i$$

If i be greater than i', after reversing, the west end would still be the highest, and we should have instead of (2)

$$\frac{\omega' - e'}{2} = i - i' \qquad (3)$$

Adding (1) and (3), there results after dividing by 2

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* The glass tube is a portion of a circle of large radius, so that the movement of the bubble indicates the angular movement of the level. To find the angular value of a division of the level scale, place the level on the telescope of some instrument to which a large vertical circle is attached. Turn the circle till the bubble passes over a number of divisions of the scale, which will be equal to the degrees, minutes, &c., through which the circle has moved; divide this number of degrees and fractions of a degree by the number of divisions of the level scale passed over, and the quotient will be the value of one division of the scale.

$$\frac{(\omega+\omega')-(e+e')}{4}=i$$

as before.

If the east end of the supporting axis be the highest instead of the west end, as we have supposed above, a similar course of reasoning would produce the formula

$$\frac{(e+e')-(\omega+\omega')}{4}=i$$

which is the same as the above, changing e and e' for ω and ω'. Both formulas may be expressed in one, thus

$$\frac{(e+e')\sim(\omega+\omega')}{4}=i \qquad (4)$$

which is the formula always to be employed for obtaining the inclination of the supporting axis.

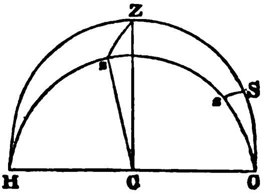
N. B. If the sum of the east readings exceed the sum of the west readings the east end of the axis is too high, and *vice versâ*.

EXAMPLE.

First position of the level	e 50	ω 40
Second " " "	e' 24	ω' 66
	$e+e'$ 74	$\omega+\omega'$ 106
		74
	$(e+e')\sim(\omega+\omega')$	32 ÷ 4
	i	8
value of 1 division of level scale		5″
	i in seconds,	40″

As a verification that the level readings have been correctly noted it may be observed that $e+\omega$ should be the same in both positions of the level, being the length of the bubble which may be supposed not to change from the effects of temperature during an observation. Thus in the above example $50+40=24+66$.

To compute now the effect of the inclination of the axis as determined above, upon the time of transit, let HZO be the vertical circle in which the telescope would play, when the supporting axis was horizontal, H*s*O the circle in which it plays when the supporting axis is inclined; then ZO*s* measured by the arc Z*s* is the inclination of the planes of these two circles, and equal



to i the inclination of the supporting axis to the horizon. The length of the arc $s_{\bullet}$ described by a star at s, in passing from the vertical to the inclined circle in which the telescope plays is expressed by $z_{x} \sin os$. (See Spher. Geom., Prop. II., Cor. 6), that is

$$s_{\bullet} = i \sin a$$

a being the altitude of the star. But the length of an arc of the equator similar to $s_{\bullet}$ is expressed by $s_{\bullet} \div \cos \delta$, δ being the star's declination. (See note on p. 153, and Spher. Geom., Prop. II. Cor. 6.) This arc of the equator is converted into time by dividing by 15. Hence

$$\frac{i \sin a}{\cos \delta} \div 15 \text{ or } \frac{i \sin a}{15 \cos \delta}$$

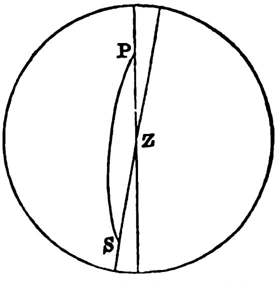
is the difference of time between the passage of the star over the middle wire of the telescope when the supporting axis is inclined, and its passage over a vertical circle. N. B. 1. The altitude of the star, designated by a in the above formula, is obtained from its declination given by catalogue, and the latitude of the place as at p. 150. 2. If the east end of the supporting axis is too high, the telescope is thrown to the west of its proper position, and the star, moving as it does from east to west by the diurnal motion, passes the wires too late; the correction therefore found as above is then subtractive. If the west end be too high, the star passes the wires too soon, and the correction is additive.

To compute the azimuth error of the instrument and its effect upon the time of transit.

This error arises from the deviation from the meridian of the vertical circle which the line of collimation describes, or to which the circle that it describes is reduced as above.

The most ready way of determining the amount of deviation and its effect on the time of transit is by "the method of high and low stars," as it is termed, that is by the transit of a star near the zenith, and one remote from it or near the horizon, whose right ascension and consequent time of transit does not differ much from the former.

To understand this let it first be supposed that one star passes the meridian exactly at the zenith, and that the other passes near the horizon, and let it be supposed also that the stars have the same right ascension. If the instrument were exactly adjusted to the meridian so that the optical axis of the telescope moved in the plane of that circle, then the two stars would be on the middle wire at the same instant, if the telescope could be brought down instantaneously from the high to the low star. But if the circle in which the telescope plays make a small angle (called the angle of deviation or azimuth error) with the meridian, then the supporting axis being supposed horizontal, when the telescope is vertical it will point to the zenith, in which all vertical circles as well as the meridian intersect, and the zenith star will be on the middle wire at the same instant as before, but the lower down the other star is, the wider will its time of transit over the middle wire differ from that of its transit over the meridian, because the farther will the vertical circle which the telescope describes be from the meridian, the farther we go from the zenith where they intersect. The greater, therefore, the difference between the time of transit of the zenith star and the low star over the wire of the instrument as compared with the difference of their times of transit over the meridian, which in the case supposed is zero, the greater the deviation of the vertical circle described by the instrument from the meridian. Now it is not necessary that the high star should pass exactly at the zenith, but only near it. If the difference of the observed time of transit of the high and low star be equal to the difference of their right ascensions, that is, of their times of passing the meridian, the optical axis of the telescope moves in the plane of the meridian; if not, this axis describes a vertical circle which deviates from the meridian, and the amount of this deviation and the consequent error in the time of meridian transit, we proceed now to show how to determine.



Let the full circle in the diagram represent the horizon, Z the zenith, P the pole, and consequently PZ the meridian. Let ZS represent the vertical circle described by the telescope, S the place of a star where it crosses it, and appears on the middle wire, and PS, the declination circle, passing through the star. In the spherical triangle PZS, in which the hour angle P represents the time that has elapsed since the star S passed the meridian, PZ, before it reached the wire of the telescope in the vertical ZS, we have by the sine proportion

$$\sin P : \sin Z :: \sin ZS : \sin PS$$

From which, taking P in place of sin P, since it is a very small angle, and Z the small angle of deviation of the vertical ZS from the meridian in place of the sine of its supplement PZS, which is also its own sine, and representing ZS by ζ and the complement of PS or the declination of the star by δ, multiplying the means of the proportion and dividing by the last term, we have

$$\text{P} = \text{Z}\,\frac{\sin \zeta}{\cos \delta^{*}} \qquad (1)$$

ζ may be found as at p. 150, by supposing S to be on the meridian, without sensible error, by means of the latitude of the station and declination of the star.

Represent the fractional part of the above formula by n. It becomes

$$\text{P} = \text{Z}n \qquad (2)$$

Suppose now another star crossing the meridian nearly at the same time. For this we have

$$\text{P}' = \text{Z}n' \qquad (3)$$

Let now t represent the observed time of transit of the later star, α its right ascension, t' and α' the same for the other star, and let e denote the error of the clock which may be supposed unknown. For the former star (since α is the time of its passing the meridian), we shall have for the value of the hour angle when it makes the transit of the middle wire,

$$\text{P} = t + e - \alpha \qquad (4)$$

And for the other star

$$\text{P}' = t' + e - \alpha' \qquad (5)$$

By subtraction of (5) from (4) the error of the clock e is eliminated, and there results

$$\text{P} - \text{P}' = (t - t') - (\alpha - \alpha') \qquad (6)$$

Substituting for P and P′ in (6) their values given by (2) and (3), (6) becomes

$$\text{Z}(n' - n) = (t - t') - (\alpha - \alpha')$$

$$\therefore \text{Z} = \frac{(t - t') - (\alpha - \alpha')}{n - n'} \qquad (7)$$

The value of the azimuth error or deviation from the meridian Z is thus found in time. To convert it into space this value must be multi-

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[*] In the later catalogues of stars, their north polar distances (N.P.D.) are given instead of their declinations. In the above formula sin (N.P.D.) would of course be in place of cos δ. When the declination of the star is south, we have still sin PS = cos δ, for sin $(90^{\circ} + \delta) = \sin (90^{\circ} - \delta) = \cos \delta$. (See App. Art. 13.)

plied by 15. The instrument may then be adjusted to the meridian by turning the screw of the Y, which admits of horizontal motion, and which has usually upon it a graduated arc, by means of which the movement of the instrument in azimuth is indicated.

To compute the effect of the azimuth error upon the time of transit take the value of z as given by (7) in time, and substitute it, before converting it into space, in either (2) or (3), which will give the value of P or P′ the hour angle in time, n or n' being an abstract number expressing the ratio of two trigonometrical lines.

The value of P is the correction to be applied to the observed time of transit of the later star, to obtain the time of its meridian transit. That of P′ the same for the other star.

If the deviation from the meridian be southwest and northeast, as in the diagram[*] where we have supposed the zenith to be south of the pole, the correction for a star south of the zenith will be subtractive, for one north additive,[†] unless the latter make an inferior transit or *sub polo*, in which case the motion being in the opposite direction, the correction is subtractive. This is evident from an inspection of the diagram.

The only remaining correction is for error of collimation. The line of collimation when this error exists describes a cone about the supporting axis as an axis, and the point in which it pierces the surface of the celestial sphere, describes a small circle of that sphere parallel to the meridian, and at a very short distance from it. The distances between these circles measured on parallels of declination may be considered every where the same without sensible error, and the time of traversing this distance by any star will be inversely as the cosine of the star's declination.

The equatorial interval between the circles in question may be found by moving the instrument in azimuth, after reversing, by means of the screw in that Y which gives horizontal motion, till the intersection of the wires is brought back to the terrestrial point on which it was placed before reversing.

The degrees and fractions of a degree passed over on the graduated arc on the Y will indicate double the error of collimation, which, divided by 15, will give the equatorial value of it in time. This multiplied by the cosine of any star's declination will give the effect of the error of collimation on the time of the star's transit.

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[*] Which will evidently be indicated by the value of z being positive. If the deviation be S.E. and N.W. then z is negative. This may be seen by trying various cases by the diagram, such as one star passing, 1st, N. of the zenith, 2nd, *sub polo*, &c., first writing the numerator in the value of z in (7) under the form $(t - a) - (t' - a')$.

[†] The only difference in the above diagram for a star north of the zenith would be that the angle of deviation itself instead of its supplement would be the angle of the triangle, but the proportion would be the same.

The error of collimation is best measured by means of a movable vertical wire, to which motion is given by a micrometer screw, as described in another place.

Should no distant terrestrial object be visible from an observatory, owing to intervening objects near at hand, a small telescope in the building having its object glass turned towards that of the transit instrument may serve as a collimator. The rays of light proceeding from the wires at the focus of the object glass of the small telescope strike this object glass, are refracted by it, and emerge in parallel lines; they then strike the object glass of the transit instrument, and are conveyed to the focus of parallel rays, which is the astronomical focus; so that in looking through the eye end of the transit instrument the wires of the small telescope will be distinctly seen. Care should be taken to throw the light of a window or lamp in at the eye end of the small telescope. A similar contrivance may be employed for a meridian mark.

But the transit instrument may be made its own collimator, by placing a vessel of mercury underneath, and turning the object end of the telescope downwards. If the axis be horizontal, and the instrument truly collimated, the wires being illuminated by an orifice in the side of the eye piece, the rays of light will pass from them to the object glass, emerge in parallel lines, strike the surface of the mercury vertically, be reflected back in the same lines, and converge to the focus of the object glass at the same points which they left, so that the reflected image of the wires will be seen coinciding with the direct image. If not, there is either error of collimation or of level, or both. If the axis had previously been made horizontal by the striding level, it is the latter, and the diaphragm containing the wires must be moved till there is coincidence between their direct and reflected images; or a movable wire may serve to measure the interval between them. This interval is double the collimation error, because the angle of incidence is equal to the angle of reflection, the former being on one side the vertical, the latter on the other. If, therefore, the direct image of the wire be brought to the vertical by the screws of the diaphragm by a movement over half the distance between the direct and reflected image, the reflected image will be brought there too.

The striding level need not be used at all, if the instrument be reversed in the Y^s, in using the collimating eye piece with a basin of mercury; for in one position of the instrument the angle obtained by taking half the distance between the direct and reflected image of the wires is the sum, and in the reverse position is the difference of level error and error of collimation. The well-known algebraic formula, "to half the sum add half the difference for the greater of the two quantities, and from half the sum subtract half the difference for the less," will serve to determine those two errors separately. To know which is the greater, the level or collimation error, we have this rule:—If the reflected image in both positions appears on the same side of the direct, then the level error is the greater of the two, but if on different sides, the collimation error is the greater. All this will appear evident if the student make a diagram with a line to represent the supporting axis with level error exaggerated, a line perpendicular to this at the middle, to represent true line of collimation, another line from the same middle point oblique to represent the erroneous line of collimation, a horizontal line below to represent the surface of the mercury, and from the point where the erroneous line of collimation meets it, a vertical and also a line making the same angle with it as does the erroneous line of collimation. In reversing the instrument the only change will be in the erroneous line of collimation, which will now make the same angle on the other side of the true.

The following example will serve to illustrate all the foregoing rules for applying the corrections to an observation with the transit instrument.

STATION AND DATE.	COLLEGE OBSERVATORY.	MARCH 8TH, 1850.
OBSERVER'S NAME.	B.	B.
STAR'S NAME.	δ GEMINORUM.	α CANIS MINORIS.
Wires. I.	7^h 11^m 11^s·5	7^h 31^m 32^s·2
II.	7 11 30	7 31 49·5
III.	7 11 48·8	7 32 7
IV.	7 12 7·5	7 32 24·4
V.	7 12 26	7 32 41·7
SUM.	5)35 59 03·8	37 40 34·8
Mean of Wires.	7 11 48·76	7 32 06·96
Corr. for Collim. Error.*	+ ·05402	+ ·05024
Corr. for Level Error.†	+ ·092	+ ·07397
Corr. for Azim. Error.‡	— 2·50	— 1·865
Time of Mer. Transit.	7 11 46·406	7 32 05·219
Star's R. Ascen.	7 11 10·20	7 31 27·38
Error of Clock.	Secs. 36·21‖	Secs. 37·839‖

* Collimation Error (δ Geminorum) thus obtained :

Equatorial Error	0·5	log.	$\bar{2}$·69897
Dec. N. Alm. March 8th, 22° 15′ 4″·9		log. cos	9·96639
Collimation Error	·05402	log.	$\bar{2}$·73258

Collimation Error (α Canis Minoris) :

Equatorial Error	·05	log.	$\bar{2}$·69897
Dec. March 8th, 5° 36′ 6″·5		log. cos	9·99792
	·05024	log.	$\bar{2}$·70105

‖ Mean of the two stars 37^s.

For Notes † and ‡ see next page.

† The level error is computed as follows:

e	96	ω	139
e'	130	ω'	105
$e+e'$	226	$\omega+\omega'$	244

$$\frac{e+e' \sim \omega+\omega'}{4}=4{\cdot}5$$

Multiplying this result by •3″ the known value of a division of the level scale, we have 1″•35 as the value of i the inclination of the supporting axis.

δ GEMINORUM.

Dec.	22° 15′ 5″	ar. co. log. cos	0•03360
Lat. of Station,	40 43		
Zenith dist.	18 27 55	log. cos	9•97704
i	1″•35	log.	0•13033
15		ar. co. log. cos	8•82391
•09223		log.	$\bar{2}$•96488

α CANIS MINORIS.

Dec.	5° 36′ 6″	ar. co. log. cos	0•00208
Lat. of Station,	40 43		
Zenith dist.	35 6 54	log. cos	9•91275
i			0•13033
			8•82391
•07397		log.	$\bar{2}$•86907

‡ Azimuth Error obtained by formulas

$$z=\frac{(t-t')-(a-a')}{n-n'}, p=z\,n, n=\frac{\sin\zeta}{\cos\delta}$$

t	obs'd time transit α Canis Minoris	7ʰ 32ᵐ 6ˢ•96	
t'	" " " δ Geminorum,	7 11 48•76	
$t-t'$	" " "		20ᵐ 18ˢ•20
a	right asc. α Canis Minoris,	7 31 27•38	
a'	" " δ Geminorum,	7 11 10•20	
$a-a'$			20ᵐ 17ˢ•18
$(t-t')-(a-a')$			1ˢ•02

α CANIS MINORIS.

Zenith dist.	35° 6′ 53″•5	log. sin	9•75983
Decli.	5 36 6 •5	log. cos	9•99792
n	•5779		$\bar{1}$•76191

δ GEMINORUM.

Zenith dist.	18° 27′ 55″•1	log. sin	9•50069
Decli.	22 15 4 •9	log. cos	9•96639
n'	•3422		$\bar{1}$•53430

The correct siderial time being ascertained by the transit of a star of known right ascension, the correct mean solar time may be found from this, as follows:

The Nautical Almanac gives on p. XXII. of each month the mean time of transit of the first point of Aries, which is the zero of siderial time, and may be called from analogy the siderial noon.* By means of this and the table of time equivalents, explained at p. 147, the mean solar time corresponding to any given siderial time, may be obtained by the following rule.

Mean solar time required = mean time at *preceding* siderial noon + the equivalent to the *given* siderial time.

EXAMPLE.

To convert 7^h 11^m $10^s{\cdot}20$ siderial time (the true time of meridian transit recorded above) March 8th, 1850, into mean solar time for the meridian of New York.

Mean time at preceding siderial noon, viz. March 8th,					0^h	56^m	$30^s{\cdot}74$
For given siderial time.	7^h	0^m	0^s	Corresponding mean solar intervals by Tab. time equivalents, p. 586, N. A.	6	58	51·19
		11	0			10	58·20
			10				9·97
			0·20				0·20
Sum mean solar time required					8^h	6^m	$30^s{\cdot}30$
Correction for longitude of N. Y.†							$48^s{\cdot}68$
					8	5	41·62

$n - n'$ ·2357 log. $\bar{1}{\cdot}37236$
$(t - t') - (a - a')$ 1·02 log. 0·00860

z. in time, $4^s{\cdot}328$ log. 0·63624
15 log. 1·17609

z. in space, $6''{\cdot}491$ log. 1·81233

z. 4·328 log. 0·63624 log. 0·63624
n. ·5779 log. $\bar{1}{\cdot}76191$ n' ·3422 log. $\bar{1}{\cdot}53430$

P. $2^s{\cdot}501$ log. 0·39815 P. 1·864 log. 0·27054

* To obtain the mean time of siderial noon at any place having a different longitude from Greenwich, it is necessary to subtract $9^s{\cdot}8565$ multiplied by the hours and fractions of an hour, by which the place differs in long. from Greenwich if the place be west, and to add this product if the place be east of Greenwich. As the daily gain of siderial time on solar is about 3^m 56^s, this divided by 24 or $9^s{\cdot}8565$ will be the hourly gain or the hourly motion of the sun backward from west to east.

† This is obtained by multiplying $9^s{\cdot}856$ by $4^h{\cdot}94$ the difference of longitude between New York and Greenwich. This correction may be applied here instead

The mean solar time may be obtained by direct observation of a meridian transit of the sun, which is made by taking the transit of each limb of the sun, that is to say observing the times when the sun's disc is tangent to the wires, both upon its western and eastern side, and taking a mean of the times as the time of transit of the sun's centre.* Or more accurately by applying as a correction additively or subtractively to the observed time of transit of the limb the time occupied by the sun's semi-diameter in passing the meridian, which is given for every day in the year in the Nautical Almanac, p. I. of each month.

To convert mean solar time into siderial the rule is as follows.

Siderial time *required* = siderial time at *preceding* mean moon + the equivalents to the *given* mean time.

EXAMPLE.

To convert 8^h 5^m $41^s{\cdot}62$ mean time at New York, into siderial time.

Sider. time at preced. mean noon, Gr. viz., March 8th,			23^h	3^m	$19^s{\cdot}98$
Correction for long. N. Y.				+	48·68
			23	4	8·66
For mean intervals.	8^h 0^m 0^s	The tab. p. 518, N. A. gives the equiv. sid. intervals.	8	1	18·85
	5 0			5	0·82
	41				41·11
	0·62				0·62
Sum sid. time required			7^h	11^m	$10^s{\cdot}06$†

The reasons for the above rules are sufficiently evident.

PROBLEM.

93. Given the latitude of the place, and the declination of a heavenly body, to determine its altitude and azimuth when on the six o'clock hour circle.

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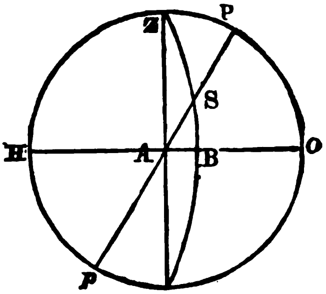
of to the mean time of siderial noon. Strictly the equivalent of $4^s{\cdot}863$ in solar intervals should be applied, or $48^s{\cdot}68$ may be applied to the given siderial time additively before taking out the solar equivalents.

* A colored glass over the eye piece is necessary in observing the sun.

† The sum amounting to more than 24^h, of course 24^h must be rejected from it, as after reaching 24^h the siderial time begins at zero again.



Let HZPO be the meridian of the place, Z the zenith, HO the horizon, S the place of the heavenly body on the six o'clock hour circle PS*p*, and ZSB the vertical circle passing through it. Then, in the right-angled triangle SBA are known AS,* the declination, and the angle SAB, or arc OP, the latitude of the place, to find the altitude BS, and AB or the complement of the azimuth OB.

EXAMPLE.

1. Required the altitude and azimuth of Arcturus when upon the six o'clock hour circle of New York, lat. 40° 43′ N., on the 1st of Jan. 1850; its declination on that day being 19° 57′ 56″ N.

By Napier's rules we have

$$\text{Rad.} \sin BS = \sin A \sin AS$$

$$\text{Rad.} \cos A = \tan AB \cot AS \therefore \cot BO = \frac{\text{Rad.} \cos A}{\cot AS}$$

	40°	43′		log. sin	9·81446	log. cos	9·87964
	19	57	56	log. sin	9·53334	log. cot	10·43974
alt.	12	52	12	log. sin	9·34780		
az.	74	38	22			log. cot	9·43990

94. There remains one case in the solution of oblique angled spherical triangles, which we have deferred to this place because we wished to employ in it the rules for the solution of right angled triangles.

This is where two sides and the angle opposite† to one of them, or two angles and the side opposite to one of them, are given; or, as it is sometimes expressed, where two of the given parts are a side and its opposite angle. In such a case we may proceed as follows:

By means of the proportion, *the sines of the angles are as the sines of the opposite sides* (Art. 81), the unknown part opposite one of the given parts may be found. Four parts of the triangle will then be known, and two will remain unknown; these two will be a side and its opposite angle, to find which, from the vertex of the unknown angle let fall an arc perpendicular upon the unknown side opposite, and the given triangle will be divided into two partial triangles, which will be right angled, and in each of which two parts will be known. Applying Napier's rules to the solution of these, the partial angles which compose the unknown angle may be found, and their sum will be the value of the unknown angle; then the unknown side opposite may be found by the proportion, the sines of the angles are as the opposite sides; or this last side may be found by calculating the two parts of which it is composed from the right angled triangles, and adding them together, which is the better method, since it avoids ambiguity. If the perpendicular arc drawn from the vertex of the unknown angle to the unknown side falls without the triangle, of course the difference, instead of the sum of the angles and sides found in the right angled triangles, is to be taken.

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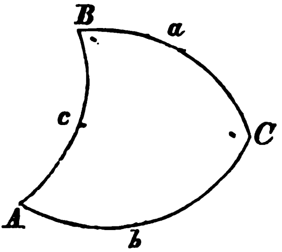
* The horizon and equator intersect at A, 90° from the meridian.

† The term opposite is used in a more exact sense here than in Napier's rules, of which we have just been speaking.



Thus in the annexed diagram triangle let the side and angle opposite given be c and C, and the other given angle B. Then first

$$\sin \text{C} : \sin c :: \sin \text{B} : \sin b$$

by means of which proportion b may be calculated and will be legitimately ambiguous. Then there will be left unknown A and a. From A let fall a perpendicular AD upon a, which we have not drawn, lest it should confuse the diagram, but which the student can imagine; then in the right angled triangle B A D we know two parts B and c, and also in the right angled triangle C A D we know two parts C and b, the latter having been found by the proportion above. To calculate the partial angles at A, calling that in the first right-angled triangle above mentioned ω, and that in the second ω', we have by Napier's rules

$$\text{R} \cos c = \cot \text{B} \cot \omega$$

whence

$$\cot \omega = \frac{\text{R} \cos c}{\cot \text{B}}$$

and in the same manner

$$\cot \omega' = \frac{\text{R} \cos b}{\cos \text{C}}$$

then

$$\omega + \omega' = \text{A}$$

and

$$\sin \text{C} : \sin c :: \sin \text{A} : \sin a$$

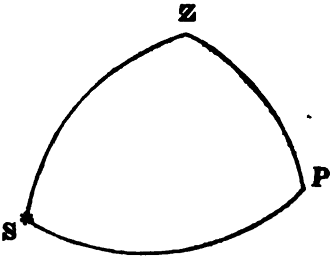
whence

$$\sin a = \frac{\sin c \sin A}{\sin C}$$

Thus all the parts of the triangle are determined.

For an application of this case of solution take the following

EXAMPLE.



Given the zenith distance, azimuth and polar distance, to find the hour angle and colatitude of the station. Or in the diagram given, ZS, Z, and PS, to find P and PZ.

Supposing the declination of a star, as given by catalogue, to be 16° 11′ N., and its observed altitude and azimuth to be 39° 10′ and 75° 10′ from the south, required the hour angle of the star and latitude of the station.

PS = 90°—16° 11′ = 73° 49′	a. c. log. sin 0·01756		log. cot 9·46271
Z = 180°—75° 10′ = 104° 50′	log. sin 9·98528	log. cos 9·40825	
ZS = 90°—39° 10′ = 50° 50′	log. sin 9·88948	log. cot 9·91095	
P = 3ʰ 25ᵐ 12ˢ = 51° 18′	log. sin 9·89232		log. cos 9·79605
17° 26′ 46″		log. tan 9·49730	
65° 6′ 6″			log. tan 10·33334
PZ = 47 39 20″			
lat. = 42 20 40			

We have now demonstrated formulas for the solution of every possible case of plane and spherical triangles, including the more simple formulas which apply exclusively to the right angled triangles.

The examination questions which follow call attention to the most important results of the investigations in the preceding pages. After these, in Appendix II. will be found many useful matters connected with Spherical Trigonometry, for which it was thought not best to interrupt the general train by which the solutions of triangles are deduced.

Note.—In assuming hypothetical cases, care must be taken not to suppose such as are impossible. The following are the governing principles to be observed.

In plane triangles, 1. One side must be less than the sum of the other two (Geom. Ax. 13 Cor.) 2. The greater side of a triangle is opposite to the greater angle (Geom. Th. 9). 3. The sum of the angles must be exactly two right angles.

In spherical triangles, the first two principles also apply (Spher. Geom., App. III. p. 2, and Prop. 7). 4. The sum of the three angles must not be less than two, nor greater than six right angles (Spher. Geom., Prop. 14). 5. The sum of the three sides must be less than a circumference (Spher. Geom., Prop. 8). 6. Each side must be less than a semicircumference (Spher. Geom., Prop. 8, Note). 7. Each angle must be less than two right angles (Spher. Geom., Prop. 8, Note).

EXAMINATION QUESTIONS IN TRIGONOMETRY.

What is the object of Trigonometry ?

How many elements are there in a triangle, and what are they ?

How many elements must be given in order to determine the rest ?

In plane triangles what must one element always be ? Why ?

What is the difference between Geometrical and Trigonometrical solutions ?

Which are most accurate ?

Are trigonometrical solutions perfectly accurate ?

Whence arises the very small inaccuracy ? *Ans.* From the decimals neglected in calculating tables of logarithms.

How is the circumference of a circle divided for the purposes of Trigonometry ?

What is the complement of an angle or arc ?

What is the supplement ?

What are complements of each other in a right angled triangle ?

What is the sine of an arc ?

What is the cosine ?

The tangent ? Cotangent ? Secant ? Cosecant ?

What trigonometrical line changes its sign with the sine ? *Ans.* The cosecant.

In which quadrant are they negative ?

What changes with the cosine ? *Ans.* The secant.

Where are they negative ?

What with the tangent ? *Ans.* The cotangent.

Where are they negative ?

In passing through what values do quantities generally change their signs ? *Ans.* Zero and infinity.

What is the least value of the sine ?

Where is it 0 ?

What is the greatest value of the sine ?

Where is it radius ?

How many times does it change its sign in going round the circumference ?

What is the least value of the tangent ?

Where is it 0 ?

What is its greatest value ?

Where is it infinite ?

How many times does it change in going round the circumference ?

N. B. Let these questions be repeated for the secant, cosine, cotangent, and cosecant.

To what is the sine of 45° equal ?
The tangent of 45° ? } Why ?
The sine of 30° ?

To what is the sine of a negative arc equal ?

The cosine of a negative arc ?

The Tangent ? Secant ? Cotangent ? Cosecant ?

To what is the sine of the supplement of an arc equal ?

The cosine of the supplement ?

The Tangent ? Secant ? Cotangent ? Cosecant ?

To what is the sine of 90° plus an arc equal ?

The cosine of 90° plus an arc ?

The Tangent ? Secant ? Cotangent ? Cosecant ?

What formula expresses the relation between the sine and cosine of an arc ? *Ans.* $R^2 = \sin^2 + \cos^2$.

What is the expression for the tangent in terms of the sine and cosine ? *Ans.* $\text{Tan} = \frac{\sin}{\cos}$

The expression for the secant ? *Ans.* $\frac{1}{\cos}$

For the cotangent ? *Ans.* $\frac{\cos}{\sin}$

For the cosecant ? *Ans.* $\frac{1}{\sin}$

How are the tangents of two arcs to each other ?

To what is the tangent equal in terms of the cotangent ? *Ans.* $\frac{1}{\cot}$ or the reciprocal of the cotangent.

What is the formula for the sine of the sum of two arcs ? *Ans.* $\text{Sin}\,(a + b) = \sin a \cos b + \sin b \cos a$ or the sum of the rectangles of the alternate sines and cosines.

The formula for the sine of the difference ? *Ans.* $\text{Sin}\,(a - b) = \sin a \cos b - \sin b \cos a$.

For the cosine of the sum ? *Ans.* $\text{Cos}\,(a + b) = \cos a \cos b - \sin a \sin b$ or the difference of the rectangles of the cosines and sines.

For the cosine of the difference ? *Ans.* $\text{Cos}\,(a - b) = \cos a \cos b + \sin a \sin b$.

For the sine of an arc in terms of half the arc ? *Ans.* $\text{Sin}\, a = 2 \sin \frac{1}{2} a \cos \frac{1}{2} a$, or twice the sine of half the arc into the cosine of half the arc.

From what is this formula deduced ?

The formula for the cosine in terms of half the arc ? *Ans.* $\text{Cos}\, a = \cos^2 \frac{1}{2} a - \sin^2 \frac{1}{2} a$.

Whence derived ?

The formula for the sine of half an arc. *Ans.* $\text{Sin}\, \frac{1}{2} a = \sqrt{\frac{1}{2} - \frac{1}{2} \cos a}$.

For the cosine of half an arc ? *Ans.* $\text{Cos}\, \frac{1}{2} a = \sqrt{\frac{1}{2} + \frac{1}{2} \cos a}$.

For the sum of the sines ? *Ans.* $\text{Sin}\, p + \sin q = 2 \sin \frac{1}{2} (p + q) \cos \frac{1}{2} (p - q)$, or twice the sine of half the sum into the cosine of half the difference.

The difference of the sines? *Ans.* Sin $p - \sin q = 2 \cos \frac{1}{2}(p+q) \sin \frac{1}{2}(p-q)$.

The sum of the cosines? *Ans.* Cos $p + \cos q = 2 \cos \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)$.

The difference of the cosines? *Ans.* Cos $p - \cos q = 2 \sin \frac{1}{2}(q+p) \sin \frac{1}{2}(q-p)$.

What is the ratio of the sum of the sines to the difference of the sines? *Ans.* $\dfrac{\tan \frac{1}{2}(p+q)}{\tan \frac{1}{2}(p-q)}$ or tan of half the sum to tan of half the difference.

How derived?

Of the sum of the sines to the sum of the cosines?

Of the difference of the sines to the sum of the cosines?

Of the sine of the sum to the sum of the sines?

Ans. $$\frac{\sin(p+q)}{\sin p + \sin q} = \frac{\frac{2}{\text{R}} \sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p+q)}{\frac{2}{\text{R}} \sin \frac{1}{2}(p+q) \cos \frac{1}{2}(p-q)} = \frac{\cos \frac{1}{2}(p+q)}{\cos \frac{1}{2}(p-q)}$$

Of the sine of the sum to the difference of the sines? *Ans.* $\dfrac{\sin \frac{1}{2}(p+q)}{\sin \frac{1}{2}(p-q)}$

What is the formula for the sine in terms of the tangent? *Ans.* $\sin a = \dfrac{\tan a}{\sqrt{1+\tan^2 a}}$

What is the formula for the tangent of the sum of two arcs? *Ans.* $\tan\,(a+b)$ $\dfrac{\tan a + \tan b}{1 - \tan a \tan b}$

For the tangent of the difference? *Ans.* $\tan\,(a-b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$

From the tangent of the sum how is the tangent of twice an arc found? Of three times an arc?

RESOLUTION OF RIGHT ANGLED PLANE TRIANGLES.

What are the three formulas for the solution of right angled triangles?

Ans. (1) Radius : the hypothenuse : : sine of one of the acute angles : the side opposite : : cosine : the side adjacent ; or radius being unity, hypothenuse × sine of either acute angle = side opposite and hyp. × cos of either angle = side adjacent.

(2) R : either of the perpendicular sides : : tangent of the angle adjacent or cotangent of the angle opposite : the other side, or R being 1, one perp. side × tan of adjacent angle = side opp.

(3) Square of the hypothenuse = sum of the squares of the other two sides. Square of either perp. side = rectangle of sum and dif. of the other two sides.

In a right angled triangle how many elements must be given? *Ans.* Two.

Why should each required element be found in terms of the two given?

In finding each unknown element how many logarithms will be employed?

When the logarithms are added what must be rejected from their sum?

When one is subtracted from the other what must be added to the latter?

When the hypothenuse is given or required with an angle, which formula is employed?

When the hypothenuse is neither given nor required?

When two sides are given to find the third?

RESOLUTION OF PLANE TRIANGLES IN GENERAL.

Two sides and the included angle of a triangle being given, how are the other elements determined? *Ans.* $a+b : a-b :: \tan \frac{1}{2}(A+B) : \tan \frac{1}{2}(A-B)$

$A+B = 180^\circ - C$

$\frac{1}{2}(A+B) + \frac{1}{2}(A-B) = A,\ \frac{1}{2}(A+B) - \frac{1}{2}(A-B) = B$

$\text{Sin } A : a :: \sin C : c.$

The three sides being given?

Ans. $\text{Sin } \frac{1}{2} A = \sqrt{\frac{(\frac{1}{2}s-b)(\frac{1}{2}s-c)}{bc}}$ or $\cos \frac{1}{2} A = \sqrt{\frac{\frac{1}{2}s(\frac{1}{2}s-a)}{bc.}}$

Two angles and the interjacent side being given? *Ans.* $180^\circ - (A+B) = C.$ $\sin C : c :: \sin A : a :: \sin B : b$

Two of the three given parts being a side and its opposite angle? *Ans.* By the sine proportion, or sines of the angles are as the opposite sides.

What is the formula for the cosine of an angle in terms of the three sides of a plane triangle? *Ans.* $\text{Cos } A = \frac{b^2+c^2-a^2}{2bc}$ or the sum of the squares of the sides which contain it, minus the square of the opposite side, divided by twice the rectangle of the containing sides.

Which is the fundamental formula in Spherical Trigonometry?

Ans. $\text{Cos } A = \frac{R^2 \cos a - R \cos b \cos c}{\sin b \sin c}$[*]

SOLUTION OF RIGHT ANGLED SPHERICAL TRIANGLES.

Upon what are Napier's rules founded? *Ans.* Upon the formulas in right angled spherical trigonometry.

How many parts are considered for the application of his rules?

What are they? *Ans.* The base, perpendicular, the complement of the hypothenuse, and the complements of the two oblique angles.

What are the rules? *Ans.* Sin of the middle part = product of the cosines of the opposite parts = product of the tangents of the adjacent parts.

N. B. Radius must be introduced homogeneously.

When two parts are given how are the rules applied?

SOLUTION OF SPHERICAL TRIANGLES IN GENERAL.

Three sides of a spherical triangle being given how are the three angles found?

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[*] All the formulas for the solution of spherical triangles may be derived from this; for applied to the three angles it gives three equations containing the six elements of the triangle from which any two elements being eliminated, an equation results containing the other four elements.

Ans. By the formulas $\sin \frac{1}{2} A = \sqrt{\frac{\sin (\frac{1}{2} s - b) \sin (\frac{1}{2} s - c)}{\sin b \sin c}}$

or $\cos \frac{1}{2} A = \sqrt{\frac{\sin \frac{1}{2} s \sin (\frac{1}{2} s - b)}{\sin b \sin c}}$

Why is not the formula for the cosine of an angle in terms of the three sides suitable for the application of logarithms?

The three angles being given how are the three sides found?

Ans. $\text{Sin} \frac{1}{2} a = R \sqrt{\frac{\cos \frac{1}{2} S \cos (\frac{1}{2} S - A)}{\sin B \sin C}}$

Two sides and the included angle being given how are the other parts found? *Ans.* By Napier's analogies. $\text{Cos} \frac{1}{2} (a + b) : \cos \frac{1}{2} (a - b) : : \cot \frac{1}{2} C : \tan \frac{1}{2} (A + B)$, $\sin \frac{1}{2} (a + b) : \sin \frac{1}{2} (a - b) : : \cot \frac{1}{2} C : \tan \frac{1}{2} (A - B)$. And then $\frac{1}{2} (A + B) + \frac{1}{2} (A - B) = A$ and $\frac{1}{2} (A + B) - \frac{1}{2} (A - B) = B$, and finally the sine proportion $\sin A : \sin B : : \sin C : \sin c$.

Two angles and the interjacent side being given? *Ans.* Napier's Analogies, 2d set.

$\text{Cos} \frac{1}{2} (A + B) : \cos \frac{1}{2} (A - B) : : \tan \frac{1}{2} c : \tan \frac{1}{2} (a + b)$

$\text{Sin} \frac{1}{2} (A + B) : \sin \frac{1}{2} (A - B) : : \tan \frac{1}{2} c : \tan \frac{1}{2} (a - b)$

$\frac{1}{2} (a + b) + \frac{1}{2} (a - b) = a$ and $\frac{1}{2} (a + b) - \frac{1}{2} (a - b) = b$, $\sin a : \sin A : : \sin c : \sin C$.

Two of the three given parts being a side and its opposite angle? *Ans.* From the vertex of that unknown angle which is opposite the unknown side, let fall a perpendicular upon this side and apply Napier's rules to the two right angled triangles thus formed. The parts of the given triangle are by this means found either directly, or by adding the parts of the two right angled triangles together.

QUESTIONS ON LOGARITHMS.

What is a logarithm?

What is the constant number which is raised to a power called?

To what is the logarithm of the base equal?

To what is the logarithm of unity equal?

What is the base of the common system?

In the common system what is the logarithm of 100? Of 1000?

Of all numbers between 100 and 1000?

Of all numbers between 1000 and 10,000?

What is the entire part of a logarithm called?

How does it compare with the number of digits in the number to which the logarithm belongs?

How is the logarithm of a number consisting of three figures found from the tables?

Of one of four?

Of one of more than four?

By the tables of Callet?

How is the number corresponding to any given logarithm found from the tables?

What is the rule for multiplication by logarithms?

For division?

Raising of powers?

Extraction of roots?

How is the logarithmic sine, tangent, &c., of any arc found when consisting of degrees and minutes only?

How the log. sine, tangent, and secant, when of seconds also?

How the cosine, cotangent, and cosecant in the last case?

How are the logarithmic secant and cosecant computed from the logarithmic sine and cosine?

What is the arithmetical complement of a logarithm?

For logarithms entering in what way into formulas are arith. comp^s. used?

What must be rejected from the logarithmic sum for each ar. comp. used?

QUESTIONS ON THE CIRCLES OF THE CELESTIAL SPHERE.

What is the axis of the earth?

The axis of the heavens?

What is the celestial equator? The ecliptic?

What are the equinoxes?

What are declination circles?

What is the meridian of a place?

What is the hour angle of a heavenly body?

What is the horizon of a place?

What are the poles of the horizon called?

What are vertical circles?

Which is called the prime vertical?

What is the declination of a heavenly body? The right ascension?

What is the celestial latitude? Longitude?

What is the altitude of a heavenly body? The azimuth?

What are the co-ordinates of a heavenly body?

How many sets are there?

Which are obtained from observation?

Which of the observed co-ordinates are preferable, and why? *Ans.* R. A, and D, because they are the same for every place on the earth, whereas altitude and azimuth are different for every place.

QUESTIONS ON THE TRANSIT INSTRUMENT.

What are the different kinds of time?

What is apparent solar time? Mean solar time?

What is siderial time?

How much longer is a mean solar than a siderial day?

How is a mean solar interval of time converted into a siderial interval, and the contrary?

What instrument is employed for observing the time of transit of stars over the meridian?

Of what parts does the transit instrument consist?

How many and what are the adjustments?

How is the instrument collimated?

How is the striding level adjusted ?

How is the instrument adjusted to the meridian ?

When the instrument is completely adjusted, in what plane does the line of collimation move ?

How is an observation made with the transit instrument ?

How is the equatorial interval of the wires determined ?

How the interval between any wire and the middle wire for a particular star ?

What is the use of knowing the intervals of the wires ?

How does the probable error of observation compare with the number of wires observed upon ?

What is the formula for the inclination of the supporting axis ?

What the formula for the correction of the time of meridian transit for level error ?

What the formula for determining error in azimuth or deviation from the meridian ?

What for consequent error in the time of meridian transit ?

How is the equatorial error of collimation found ?

How from this the collimation error for any star ?

What is used with the transit instrument ?

If the clock keep true siderial time, what does the time of meridian transit show ?

If the right ascension of the object observed is known by catalogue, what does the difference between this and the time of meridian transit show ?

How is siderial time converted into solar and the converse ?*

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* The above questions will suffice to show the nature of those which should be put upon the subsequent parts of the work with which we shall not take up further space.

APPENDIX II.

ON UNLIMITED SPHERICAL TRIANGLES AND THEIR SOLUTION.*

OF THE VARIOUS TRIANGLES FORMED BY THE SAME THREE POINTS ON THE SPHERE.

1. If any two points, A and B, be taken upon the surface of the sphere, the arc of a great circle joining them may be considered to be either the arc A B ($< 180^\circ$), or $360^\circ - $ A B; or if we do not limit the arcs to values less than a circumference, we may consider it to have an indefinite number of values expressed generally by the formula $2\,n\,\pi \pm a$, a denoting that value which is less than π or a semicircumference, and n any whole number or zero.

2. If two arcs of great circles intersect in a point A, the angle which they form may be considered to be either the angle A ($< 90^\circ$), or $180^\circ - $ A, or $180^\circ +$ A, or $360^\circ - $ A; or, taking the most general view of angular magnitude, the angle will have an indefinite number of values expressed by the formula $m\,\pi \pm A$, A denoting the value which is less than $\frac{1}{2}\,\pi$, and m any whole number or zero.

3. If, therefore, any three points, A, B, C, be taken on the surface of the sphere, and great circles, made to pass through each pair, we shall have an infinite series of triangles whose sides will be generally expressed by

$$2\,n\,\pi \pm a,\quad 2\,n\,\pi \pm b,\quad 2\,n\,\pi \pm c \qquad (1)$$

and whose angles will be generally expressed by

$$m\,\pi \pm A,\quad m\,\pi \pm B,\quad m\,\pi \pm C; \qquad (2)$$

a, b, c, denoting the arcs less than π joining the pairs of points B C, A C, A B, respectively; A, B, C, the angles less than $\frac{1}{2}\,\pi$ formed at those points by the intersection of these arcs; and n and m, any whole numbers, or zero.

4. It is evident, however, that we cannot assume that any three values of the sides from the series (1), combined with any three values of the angles from (2), will form a spherical triangle. Some general relations of the parts composing a triangle must first be established, from which corresponding values of n and m in (1) and (2)

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* Introduced by Gauss. Notwithstanding the elegance and generality thus given, to the solutions of many astronomical problems, nothing is to be found on this subject in our trigonometrical works. The present paper is from Prof. Chauvenet of the U. S. Naval Acad. The explanatory notes are the author's.

may be deduced. Although these general relations are well known, it may not be out of place to add here a concise demonstration of them.

Let the point C,[*] one of the angular points of the spherical triangle ABC, be referred by rectangular co-ordinates to three planes, one of which, the plane of $x\,y$, is the plane of the great circle A B; let the axis of x be the diameter of the sphere passing through B, and let the origin be the centre of the sphere. The formulas of transformation from these co-ordinates to polar co-ordinates, the origin being the same, the polar axis being the axis of x, and the fixed plane the plane of $x\,y$, are

$$\left.\begin{aligned} x &= \text{R} \cos a \\ y &= \text{R} \sin a \cos \text{B} \\ z &= \text{R} \sin a \sin \text{B} \end{aligned}\right\} \qquad (3)$$

where B denotes the angle which the plane passing through the polar axis and the point C makes with the fixed plane; R, the radius-vector in this plane, or distance of the point C from the origin; and a the angle which this radius-vector makes with the polar axis. B is an angle of the spherical triangle, and a is the side opposite the angle A; and, according to the principles of analytical geometry, B and a may be altogether unlimited, due regard being had to the signs of their trigonometric functions, and to those of x, y, and z.

Let us now transform from these rectangular co-ordinates to others also rectangular, the origin and the plane of $x\,y$ remaining the same, but the axis of x in the new system passing through the point A, and therefore making with the first axis the angle c, c also expressing the side of the triangle opposite the angle C. The known formulas of transformation become

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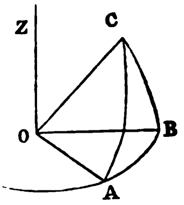
[*] The rest of Art. 4 implies some knowledge of Analytical Geometry. It may be readily understood, however, by the mere student of trigonometry from the annexed diagram, with the following explanations. R in formulas (3) is equal to O C in the diagram. The projection of R or of O C, on O B which is called the axis of x, that is to say the distance between the foot of a perpendicular from C on O B and the point O is the value of x in the formulas, the projection of O C on a line called the axis of Y, drawn from O in the plane A O B perpendicular to O B, is the value of y in the formulas, and the projection of R or O C on O Z perpendicular to the plane A O B, is the value of z in the formulas. The first of formulas (3) is now obvious enough; in the second R sin a evidently expresses the value of a perpendicular from C to OB in the plane C O B, and this perpendicular multiplied by the cosine of the angle which it makes with its projection on the plane A O B equal to the angle of the two planes or the angle B, expresses the length of the projection on the plane A O B, which is evidently equal to the projection of R on the axis of Y; or multiplied by the sin B expresses the height of C above the plane A O B, which is equal to the projection of R on O Z the axis of Z.



N. B. The axes of x, y, and z are at right angles each to the plane of the other two. So also are those of x', y', and z'.

The student will readily deduce formulas (4) by the rules of Plane Trigonometry.

$$\left.\begin{aligned} x &= x' \cos c - y' \sin c \\ y &= x' \sin c + y' \cos c \\ z &= z' \end{aligned}\right\} \qquad (4)$$

Finally, the formulas of transformation from this last system of rectangular again to polar co-ordinates, the origin being the same, the fixed plane the plane of $x'\,y'$ and the polar axis the axis of x', are

$$\left.\begin{aligned} x' &= \mathrm{R} \cos b \\ y' &= -\mathrm{R} \sin b \cos \mathrm{A} \\ z' &= \mathrm{R} \sin b \sin \mathrm{A} \end{aligned}\right\} \qquad (5)$$

Now, if the values of x, y, z, x', y', z', given by (3) and (5) be substituted in (4), we have at once the following system of equations:—

$$\left.\begin{aligned} \cos a &= \cos c \cos b + \sin c \sin b \cos \mathrm{A} \\ \sin a \cos \mathrm{B} &= \sin c \cos b - \cos c \sin b \cos \mathrm{A} \\ \sin a \sin \mathrm{B} &= \sin b \sin \mathrm{A} \end{aligned}\right\} \qquad (6)$$

which are the known fundamental formulas of spherical trigonometry, but established without imposing any restrictions upon the values of the parts of the triangle.

From this investigation it appears that these formulas may be regarded as formulas of transformation from one system of polar co-ordinates to another, or rather from one system of spherical co-ordinates to another. For example, the co-ordinates of a star referred to the pole of the equator and the meridian of a place whose colatitude is c, are its polar distance a, and its hour angle B; the co-ordinates of the same star referred to the pole of the horizon and the meridian, are its zenith distance b, and its azimuth A; and the formulas (6) express the relations by means of which we can pass from one of these systems to the other.

(5.) Let us now inquire what are the corresponding values of the sides and angles in the series of triangles expressed by (1) and (2). Let a, b, c, A, B, C, denote the values of the parts of *one* of these triangles, which, if we please, we may suppose to be the triangle whose parts are less than π. Then since

$$\sin (2\,n\,\pi + \phi) = \sin \phi, \ \cos (2\,n\,\pi + \phi) = \cos \phi,$$

the equations (6) will be satisfied by the substitution of $2\,n\,\pi + a$, $2\,n\,\pi + b$, &c., for a, b, and c; and therefore the triangle (a, b, c, A, B, C) is the first of an infinite series obtained from it by the successive addition of $2\,\pi$ to each or all of its parts, every triangle of the series being such, that the relations of its parts are expressed by (6), when a, b, c, A, B, C, are assumed to represent those parts.

It is evident, also, from the principle of "uniformity of direction" observed in the preceding demonstration in reckoning the sides and angles, that we must be able to satisfy the equations, by making either all the sides, or all the angles, or all the sides and angles, *negative at the same time*,* and, considering each of the triangles

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* The student will do well to conceive the position of the angular points of the triangles on the surface of the sphere with these variations.

N. B. That the sides are all negative together, or the angles together, or both together.

The same thing stated in the text may be made evident by referring to equations thus obtained as the first of a series, as above, we have three more series. We have then the four series following :—

1st Series.	2d Series.
$2n\pi + a,\ 2n\pi + A$	$2n\pi - a,\ 2n\pi + A$
$2n\pi + b,\ 2n\pi + B$	$2n\pi - b,\ 2n\pi + B$
$2n\pi + c,\ 2n\pi + C$	$2n\pi - c,\ 2n\pi + C$

3d Series.	4th Series.
$2n\pi + a,\ 2n\pi - A$	$2n\pi - a,\ 2n\pi - A$
$2n\pi + b,\ 2n\pi - B$	$2n\pi - b,\ 2n\pi - B$
$2n\pi + c,\ 2n\pi - C$	$2n\pi - c,\ 2n\pi - C$

In all the terms of these series, n may have the same or different values; and we thus have all the possible combinations of the values represented by (1) and (2), *so long as* m *in* (2) *is even*. But if we substitute $2n + 1$ for m we shall find that the following series will satisfy the equations (6) :—*

5th Series.	6th Series.
$2n\pi + a,\quad 2n\pi + A$	$2n\pi - a,\quad 2n\pi + A$
$2n\pi - b,\ (2n + 1)\pi + B$	$2n\pi + b,\ (2n + 1)\pi + B$
$2n\pi - c,\ (2n + 1)\pi + C$	$2n\pi + c,\ (2n + 1)\pi + C$

7th Series.	8th Series.
$2n\pi + a,\quad 2n\pi - A$	$2n\pi - a,\quad 2n\pi - A$
$2n\pi - b,\ (2n + 1)\pi - B$	$2n\pi + b,\ (2n + 1)\pi - B$
$2n\pi - c,\ (2n + 1)\pi - C$	$2n\pi + c,\ (2n + 1)\pi - C$

the 6th, 7th, and 8th of which series are derived from the 5th, as the 2d, 3d, and 4th were derived from the 1st, in the preceding paragraph.

By successively exchanging a for b and c, we find eight more series, namely,

9th Series.	10th Series.
$2n\pi - a,\ (2n + 1)\pi + A$	$2n\pi + a,\ (2n + 1)\pi + A$
$2n\pi + b,\quad 2n\pi + B$	$2n\pi - b,\quad 2n\pi + B$
$2n\pi - c,\ (2n + 1)\pi + C$	$2n\pi + c,\ (2n + 1)\pi + C$

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(6), which involve all the relations of the six elements of a spherical triangle, and which will be satisfied by changing simultaneously a, b, and c into $-a, -b, -c$, or A, B, C, into $-A, -B, -C$ or both; observing the general rule that $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$.

* The student will try these elements given in the 5th series, in eqs. (6), observing that $\cos\{(2n + 1)\pi + \phi\} = \cos(180^\circ + \phi) = -\cos\phi$ and $\sin\{(2n + 1)\pi + \phi\} = -\sin\phi$.

11th Series.	12th Series.
$2n\pi - a$, $(2n+1)\pi - A$	$2n\pi + a$, $(2n+1)\pi - A$
$2n\pi + b$, $2n\pi - B$	$2n\pi - b$, $2n\pi - B$
$2n\pi - c$, $(2n+1)\pi - C$	$2n\pi + c$, $(2n+1)\pi - C$

13th Series.	14th Series.
$2n\pi - a$, $(2n+1)\pi + A$	$2n\pi + a$, $(2n+1)\pi + A$
$2n\pi - b$, $(2n+1)\pi + B$	$2n\pi + b$, $(2n+1)\pi + B$
$2n\pi + c$, $2n\pi + C$	$2n\pi - c$, $2n\pi + C$

15th Series.	16th Series.
$2n\pi - a$, $(2n+1)\pi - A$	$2n\pi + a$, $(2n+1)\pi - A$
$2n\pi - b$, $(2n+1)\pi - B$	$2n\pi + b$, $(2n+1)\pi - B$
$2n\pi + c$, $2n\pi - C$	$2n\pi - c$, $2n\pi - C$

5. Since three great circles by their mutual intersections (provided they have not a common diameter), divide the surface of the whole sphere into eight primitive triangles (whose parts are all less than π), the three angular points of each of which give sixteen triangles, whose parts are all less than 2π,* therefore, *three great circles of the sphere form in general one hundred and twenty-eight triangles, each of which may be considered as the first term of an infinite series of triangles formed from it by the successive addition of 2π to each or all of its parts.*

AMBIGUITY IN THE SOLUTION OF THE GENERAL SPHERICAL TRIANGLE.

For the sake of brevity, I shall call the spherical triangle, whose parts are only limited by the condition $< 360^\circ$, *the general spherical triangle.* Although any three points of the surface of the sphere may be regarded (in general) as the angular points of sixteen such triangles, yet to the problem "given three parts of the triangle to find the other three," there will in every case be but two solutions, *i. e.* two triangles containing the same data. From the equations (6), and the consequences that flow from them, we can always obtain expressions for both the sine and cosine of each of the required parts, which would fully determine the triangle, were it not that in every case one of these expressions at least involves a radical of the second degree, and has either two different numerical values, or two values numerically equal with opposite signs. To avoid this ambiguity it was thought expedient to limit all the parts of the triangles to values less than 180°, or to consider only the simple geometrical triangle. By this means all the cases in which the required quantities can be found by a cosine or tangent, without involving radicals, become fully determined. But this occurs in but four of the six cases, the other two still having two solutions; so that although six conditions were thus imposed, three limiting the data themselves, and three the quæsita, the object of removing all ambiguity was not reached.

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* *i. e.*, making $n = 0$, in each of the 16 series of the last art.

We shall see from the solutions of the general triangle, that *the ambiguity is entirely removed in every case by the imposition of a single condition restricting the sign of either the sine or cosine of but one of the required parts.* The general method here, as in many parts of the mathematics, is therefore the simplest.

FORMULAS REQUIRED FOR THE SOLUTION OF THE GENERAL SPHERICAL TRIANGLE.

1. As the formulas (6) are the same as those deduced in trigonometrical works for limited spherical triangles, we may avail ourselves, for the solution of the general triangle, of all the formulas (found in those works), deduced from them in a general manner. It is not necessary therefore to repeat all those deductions here; but I shall add a demonstration of GAUSS's equations, slightly differing from the common one, in order to establish them in their generality.

2. GAUSS's THEOREM. If

$$\begin{array}{ll} p = \cos \tfrac{1}{2} c \sin \tfrac{1}{2} (A + B), & P = \cos \tfrac{1}{2} C \cos \tfrac{1}{2} (a - b) \\ q = \cos \tfrac{1}{2} c \cos \tfrac{1}{2} (A + B), & Q = \sin \tfrac{1}{2} C \cos \tfrac{1}{2} (a + b) \\ r = \sin \tfrac{1}{2} c \sin \tfrac{1}{2} (A - B), & R = \cos \tfrac{1}{2} C \sin \tfrac{1}{2} (a - b) \\ s = \sin \tfrac{1}{2} c \cos \tfrac{1}{2} (A - B), & S = \cos \tfrac{1}{2} C \sin \tfrac{1}{2} (a + b) \end{array}$$

then the products pq, pr, ps, qr, qs, rs, *are respectively equal to the products* PQ, PR, PS, QR, QS, RS.

To demonstrate this, we have only to form the following equations, which are easily deduced from the fundamental formulas:—

$$\sin c\,(\sin A \pm \sin B) = \sin C\,(\sin a \pm \sin b)^{*}$$
$$\sin c\,(\cos A \pm \cos B) = (1 \mp \cos C) \sin (a \pm b)^{\dagger}$$
$$(1 \pm \cos c) \sin (A \pm B) = \sin C\,(\cos b \pm \cos a)^{\ddagger}$$

which, transformed by the formulas of the trigonometric analysis,§ give respectively

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* By combining $\dfrac{\sin c}{\sin C} = \dfrac{\sin a}{\sin A} = \dfrac{\sin b}{\sin B}$

† By the second of (6) we have

$$\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C$$
$$\sin c \cos B = \cos b \sin a - \sin b \cos a \sin C$$

By addition and subtraction of these we obtain

$$\sin c\,(\cos A \pm \cos B) = (\sin a \cos b \pm \sin b \cos a)\,(1 \mp \cos C)$$

whence the formula in the text.

‡ This is obtained from the last by substituting $\pi - C$ &c., for C to produce the supplemental or polar triangle.

§ To wit: formulas which express the factors of the above forms in terms of the sines and cosines of $\tfrac{1}{2} C$, $\tfrac{1}{2} c$, $\tfrac{1}{2} (A \pm B)$, $\tfrac{1}{2} (a \pm b)$ by means of which we have the following, each of the above forms furnishing two:

$$2 \sin \tfrac{1}{2} c \cos \tfrac{1}{2} c\, 2 \sin \tfrac{1}{2} (A + B) \cos \tfrac{1}{2} (A - B) = 2 \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C\, 2 \sin \tfrac{1}{2} (a + b) \cos \tfrac{1}{2} (a - b)$$

$$2 \sin \tfrac{1}{2} c \cos \tfrac{1}{2} c\, 2 \sin \tfrac{1}{2} (A - B) \cos \tfrac{1}{2} (A + B) = 2 \sin \tfrac{1}{2} C \cos \tfrac{1}{2} C\, 2 \sin \tfrac{1}{2} (a - b) \cos \tfrac{1}{2} (a + b) \text{ \&c., \&c.}$$

$$\begin{array}{ll} p\,s = \mathrm{P\,S}, & q\,r = \mathrm{Q\,R} \\ q\,s = \mathrm{Q\,S}, & p\,r = \mathrm{P\,R} \\ p\,q = \mathrm{P\,Q}, & r\,s = \mathrm{R\,S} \end{array}$$

3. ***The same notation being employed, the quantities*** p^2, q^2, r^2, s^2, ***are respectively equal to the quantities*** P^2, Q^2, R^2, S^2.

For we have $p\,q \times p\,r = \mathrm{P\,Q} \times \mathrm{P\,R}$ and $q\,r = \mathrm{Q\,R}$, whence by division $p^2 = \mathrm{P}^2$, and in the same way $q^2 = \mathrm{Q}^2$, $r^2 = \mathrm{R}^2$ $s^2 = \mathrm{S}^2$.

4. GAUSS'S EQUATIONS. From the preceding paragraph we deduce

$$\begin{array}{l} p = \pm\,\mathrm{P} \\ q = \pm\,\mathrm{Q} \\ r = \pm\,\mathrm{R} \\ s = \pm\,\mathrm{S} \end{array}$$

In these equations the positive sign must be taken in all the second members, or the negative sign in all of them.

For if we take $p = +\,\mathrm{P}$, the equations $p\,q = \mathrm{P\,Q}$, $p\,r = \mathrm{P\,R}$, $p\,s = \mathrm{P\,S}$, being divided by this, give $q = +\,\mathrm{Q}$, $r = +\,\mathrm{R}$, $s = +\,\mathrm{S}$. But if we take $p = -\,\mathrm{P}$, the same equations divided by this, give $q = -\,\mathrm{Q}$, $r = -\,\mathrm{R}$, $s = -\,\mathrm{S}$.

Hence the two following groups of equations, the first group comprising those commonly known as GAUSS'S equations, which are identical with (I.), (II.), (III.) and (IV.) of Art. 86, by clearing the latter of fractions.

$$\left.\begin{array}{l} \cos \tfrac{1}{2} c \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = \cos \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a - b) \\ \cos \tfrac{1}{2} c \cos \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = \sin \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a + b) \\ \sin \tfrac{1}{2} c \sin \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = \cos \tfrac{1}{2} \mathrm{C} \sin \tfrac{1}{2} (a - b) \\ \sin \tfrac{1}{2} c \cos \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = \sin \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a - b) \end{array}\right\} \qquad (7)$$

$$\left.\begin{array}{l} \cos \tfrac{1}{2} c \sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = -\cos \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a - b) \\ \cos \tfrac{1}{2} c \cos \tfrac{1}{2} (\mathrm{A} + \mathrm{B}) = -\sin \tfrac{1}{2} \mathrm{C} \cos \tfrac{1}{2} (a + b) \\ \sin \tfrac{1}{2} c \sin \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = -\cos \tfrac{1}{2} \mathrm{C} \sin \tfrac{1}{2} (a - b) \\ \sin \tfrac{1}{2} c \cos \tfrac{1}{2} (\mathrm{A} - \mathrm{B}) = -\sin \tfrac{1}{2} \mathrm{C} \sin \tfrac{1}{2} (a + b) \end{array}\right\} \qquad (8)$$

5. Now when the parts of the triangle are limited to values less than 180°, the second of these groups is excluded, since $\cos \tfrac{1}{2} c$, $\sin \tfrac{1}{2} (\mathrm{A} + \mathrm{B})$, $\cos \tfrac{1}{2} \mathrm{C}$, $\cos \tfrac{1}{2} (a - b)$ are then all positive. But when the triangle is unlimited, both groups must be admitted, and the question arises, when are we to employ the positive, and when the negative sign? GAUSS himself has remarked (*Theoria Mot. Corp. Cœl.*, Art. 54), that cases occur in practice in which it is necessary to employ the negative sign, and promises elsewhere a fuller explanation, which, however, I have not been able to find. But the nature of these cases and the answer to the question above propounded will be easily inferred from the following considerations.

We have seen that the formulas (6) apply not only to the triangle whose parts a, b, c, A, B, C, are all less than 360°, or 2π, but also to all the triangles whose parts are $2\,n\,\pi + a$, $2\,n\,\pi + b$, $2\,n\,\pi + c$, $2\,n\,\pi + \mathrm{A}$, $2\,n\,\pi + \mathrm{B}$ $2\,n\,\pi + \mathrm{C}$, n being

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The first two of the six that would result are all that we have thought necessary to write. They are identical with

$$p\,s = \mathrm{P\,S} \qquad q\,r = \mathrm{Q\,R}$$

any whole number or zero, and admitting of different values in each of the parts. Let us, therefore, substitute in (7) the following values of these parts:—

$$\begin{array}{ll} 2n_1\pi + a, & 2m_1\pi + \text{A} \\ 2n_2\pi + b, & 2m_2\pi + \text{B} \\ 2n_3\pi + c, & 2m_3\pi + \text{C} \end{array}$$

We shall have for the factors of the first members values similar to the following:

$$\cos\,(n_3\pi + \tfrac{1}{2}c) = (-1)^{n_3}\cos\tfrac{1}{2}c^*$$

$$\sin\,(n_3\pi + \tfrac{1}{2}c) = (-1)^{n_3}\sin\tfrac{1}{2}c^*$$

$$\cos\,[(m_1 + m_2)\pi + \tfrac{1}{2}(\text{A} + \text{B})] = (-1)^{m_1 + m_2}\cos\tfrac{1}{2}(\text{A} + \text{B})$$

$$\sin\,[(m_1 + m_2)\pi + \tfrac{1}{2}(\text{A} + \text{B})] = (-1)^{m_1 + m_2}\sin\tfrac{1}{2}(\text{A} + \text{B})$$

$$\cos\,[(m_1 - m_2)\pi + \tfrac{1}{2}(\text{A} - \text{B})] = (-1)^{m_1 - m_2}\cos\tfrac{1}{2}(\text{A} - \text{B})$$

$$\sin\,[(m_1 - m_2)\pi + \tfrac{1}{2}(\text{A} - \text{B})] = (-1)^{m_1 - m_2}\sin\tfrac{1}{2}(\text{A} - \text{B})$$

Now whatever the values of m_1 and m_2, $m_1 + m_2$ and $m_1 - m_2$ are both even or both odd at the same time, and therefore *the above substitution gives the same sign to all the first members of our equations.* In the same way it is shown that the second members will all have the same sign; and we may consequently express the result of the substitution thus:—

$$\begin{aligned} \cos\tfrac{1}{2}c\,\sin\tfrac{1}{2}(\text{A} + \text{B}) &= (-1)^n\cos\tfrac{1}{2}\text{c}\,\cos\tfrac{1}{2}(a - b) \\ \cos\tfrac{1}{2}c\,\cos\tfrac{1}{2}(\text{A} + \text{B}) &= (-1)^n\sin\tfrac{1}{2}\text{c}\,\cos\tfrac{1}{2}(a + b) \\ \sin\tfrac{1}{2}c\,\sin\tfrac{1}{2}(\text{A} - \text{B}) &= (-1)^n\cos\tfrac{1}{2}\text{c}\,\sin\tfrac{1}{2}(a - b) \\ \sin\tfrac{1}{2}c\,\cos\tfrac{1}{2}(\text{A} - \text{B}) &= (-1)^n\sin\tfrac{1}{2}\text{c}\,\sin\tfrac{1}{2}(a + b) \end{aligned}$$

which single group involves both (7) and (8). *The group* (7) *will represent one series of triangles, while the group* (8) *will represent another series, the two differing in each of their elements by some multiple of* 2π, *and the primitive triangle may belong to one or the other of these series. We may dispense, therefore, in practice, with group* (8), *by deducting* 360° *or* 2π *from the elements found by group* (7), *till they become less than* 360°, *as required for use.* (*See top of this page.*)

6. When the parts of the triangle are interchanged in GAUSS's equations, it would seem to require proof that the same sign, whether + or —, must continue in these equations; *i. e.* that when the triangle is such as to satisfy the equation,

$$\cos\tfrac{1}{2}c\,\sin\tfrac{1}{2}(\text{A} + \text{B}) = \cos\tfrac{1}{2}\text{c}\,\cos\tfrac{1}{2}(a - b)$$

* Apply (3) and (6) of Art. 70 to sin and cos of $2n_3\pi + \tfrac{1}{2}c$ to obtain these results. Or observe that $\cos\,(n\pi + \tfrac{1}{2}c) = \pm\cos\tfrac{1}{2}c$ according as n is even or odd.

it will also satisfy the equations,

$$\left.\begin{aligned}\cos\tfrac{1}{2}b\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{C}) &= \cos\tfrac{1}{2}\mathrm{B}\cos\tfrac{1}{2}(a-c)\\ \cos\tfrac{1}{2}a\sin\tfrac{1}{2}(\mathrm{B}+\mathrm{C}) &= \cos\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}(b-c)\end{aligned}\right\}\qquad(e)$$

and that when it is such as to satisfy the equation,

$$\cos\tfrac{1}{2}c\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{B}) = -\cos\tfrac{1}{2}\mathrm{C}\cos\tfrac{1}{2}(a-b)$$

it will also satisfy the equations

$$\left.\begin{aligned}\cos\tfrac{1}{2}b\sin\tfrac{1}{2}(\mathrm{A}+\mathrm{C}) &= -\cos\tfrac{1}{2}\mathrm{B}\cos\tfrac{1}{2}(a-c)\\ \cos\tfrac{1}{2}a\sin\tfrac{1}{2}(\mathrm{B}+\mathrm{C}) &= -\cos\tfrac{1}{2}\mathrm{A}\cos\tfrac{1}{2}(b-c)\end{aligned}\right\}\qquad(f)$$

To demonstrate this, we will show that the groups to which the equations (*e*) belong may be derived from (7), and those to which (*f*) belong from (8), by merely linear transformations, and therefore without again introducing the double sign. Let the equations (7) be written thus:—

$$\left.\begin{aligned}\frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\cos\frac{1}{2}\mathrm{C}} &= \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c}\\ \frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} &= \frac{\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}c}\\ \frac{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos\frac{1}{2}\mathrm{C}} &= \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}c}\\ \frac{\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\sin\frac{1}{2}\mathrm{C}} &= \frac{\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}c}\end{aligned}\right\}\qquad(g)$$

The sum and difference of the first two, and the sum and difference of the last two, give*

$$\begin{aligned}\frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}+\mathrm{C})}{\sin\mathrm{C}} &= -\frac{\sin\frac{1}{2}a\sin\frac{1}{2}b}{\cos\frac{1}{2}c}\\ \frac{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B}-\mathrm{C})}{\sin\mathrm{C}} &= \frac{\cos\frac{1}{2}a\cos\frac{1}{2}b}{\cos\frac{1}{2}c}\\ \frac{\cos\frac{1}{2}(\mathrm{A}-\mathrm{B}+\mathrm{C})}{\sin\mathrm{C}} &= \frac{\cos\frac{1}{2}a\sin\frac{1}{2}b}{\sin\frac{1}{2}c}\\ \frac{\cos\frac{1}{2}(-\mathrm{A}+\mathrm{B}+\mathrm{C})}{\sin\mathrm{C}} &= \frac{\sin\frac{1}{2}a\cos\frac{1}{2}b}{\sin\frac{1}{2}c}\end{aligned}$$

By differently combining these four equations, two and two, we may either reproduce the group (*g*) or the two groups represented by (*e*). Thus the sum and difference of the first and third, and of the second and fourth, give

$$\begin{aligned}\frac{\sin\frac{1}{2}\mathrm{B}\sin\frac{1}{2}(\mathrm{A}+\mathrm{C})}{\sin\mathrm{C}} &= \frac{\sin\frac{1}{2}b\cos\frac{1}{2}(a-c)}{\sin c}\\ \frac{\cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}(\mathrm{A}+\mathrm{C})}{\sin\mathrm{C}} &= \frac{\sin\frac{1}{2}b\cos\frac{1}{2}(a+c)}{\sin c}\\ \frac{\sin\frac{1}{2}\mathrm{B}\sin\frac{1}{2}(\mathrm{A}-\mathrm{C})}{\sin\mathrm{C}} &= \frac{\cos\frac{1}{2}b\sin\frac{1}{2}(a-c)}{\sin c}\\ \frac{\cos\frac{1}{2}\mathrm{B}\cos\frac{1}{2}(\mathrm{A}-\mathrm{C})}{\sin\mathrm{C}} &= \frac{\cos\frac{1}{2}b\sin\frac{1}{2}(a+c)}{\sin c}\end{aligned}$$

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* See the mode of deducing (I), (II), &c., of Art. 86.

These multiplied by

$$\frac{\sin \text{C}}{\sin \text{B}} = \frac{\sin c}{\sin b}$$

give

$$\frac{\sin \frac{1}{2}(\text{A}+\text{C})}{\cos \frac{1}{2}\text{B}} = \frac{\cos \frac{1}{2}(a-c)}{\cos \frac{1}{2} b}$$
$$\frac{\cos \frac{1}{2}(\text{A}+\text{C})}{\sin \frac{1}{2}\text{B}} = \frac{\cos \frac{1}{2}(a+c)}{\cos \frac{1}{2} b}$$
$$\frac{\sin \frac{1}{2}(\text{A}-\text{C})}{\cos \frac{1}{2}\text{B}} = \frac{\sin \frac{1}{2}(a-c)}{\sin \frac{1}{2} b}$$
$$\frac{\cos \frac{1}{2}(\text{A}-\text{C})}{\sin \frac{1}{2}\text{B}} = \frac{\sin \frac{1}{2}(a+c)}{\sin \frac{1}{2} b}$$

Precisely the same transformations applied to (8) would of course give a similar result with the negative sign. Hence

In the three groups which GAUSS'S *equations form by the permutation of the letters, the positive sign must be taken in all the equations, or the negative sign in all of them.*

AUXILIARY ANGLES.

It will be convenient to premise here the following proposition, upon which depends the proper employment of auxiliary angles in preparing our general formulas for logarithmic computation.

In the equations

$$\left.\begin{aligned} k \sin \phi &= m \\ k \cos \phi &= n \end{aligned}\right\} \qquad (9)$$

whatever the values of m *and* n, *we can always determine* k *and* ϕ *so as to satisfy at once these equations, and any one of the following conditions arbitrarily imposed.*

1st k positive, $(\phi < 360^\circ)$,
2d k negative, $(\phi < 360^\circ)$,
3d $\phi > 0$ and $< 180^\circ$,
4th $\phi > 180^\circ$ and $< 360^\circ$,
5th $\phi < 90^\circ$ and $> -90^\circ$,
6th $\phi > 90^\circ$ and $< 270^\circ$.

The six conditions above stated are obviously equivalent to the following; 1st, $k +$; 2d, $k -$; 3d, $\sin \phi +$; 4th, $\sin \phi -$; 5th, $\cos \phi +$; 6th, $\cos \phi -$. Of these six conditions, however, we commonly employ only the first, third, or fifth.

The quotient of the equations (9), $\tan \phi = \frac{m}{n}$, gives two values of ϕ under 360°.* Hence, also, two values of k, which will be numerically equal with opposite signs, since the two values of sin ϕ will be numerically equal with opposite signs, as also the two values of cos ϕ. If we restrict the sign of any one of the three quantities k, sin ϕ, cos ϕ, the signs of the other two will become known, and there will be but one value of k and one of ϕ under that restriction.

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* Because for any arc less than 360° there is always another arc < 360°, having the same tangent. If the former be in the first quadrant, the latter is in the 3d; if the former be in the 2d, the latter is in the 4th quadrant.

SOLUTION OF THE SEVERAL CASES OF THE GENERAL SPHERICAL TRIANGLE.

In the solutions of the various cases of spherical triangles, it is of the first importance to have simple and clear precepts, both for removing the ambiguity that occurs in every case and for determining properly the auxiliary angles. Examples might be pointed out, in recent works on trigonometry, of incorrect numerical solutions resulting from an erroneous application of precepts, in themselves correct, but not sufficiently simple or explicit. I have, therefore, given special attention to this point in arranging the following solutions. These solutions have also been carefully verified by the computation of the two triangles following:—

FIRST TRIANGLE.

$a = 125^\circ\ 0'\ 0''$ $\qquad A = 264^\circ\ 51'\ 30''.4$
$b = 140\ \ 0\ \ 0$ $\qquad B = 231\ \ 24\ \ 6\ .9$
$c = 46\ \ 0\ \ 0$ $\qquad C = 299\ \ 0\ \ 0\ .6$

SECOND TRIANGLE.

$a = 40^\circ\ 0'\ 0''$ $\qquad A = 319^\circ\ 21'\ 21''.4$
$b = 250\ \ 0\ \ 0$ $\qquad B = 107\ \ 46\ \ 57\ .6$
$c = 230\ \ 0\ \ 0$ $\qquad C = 50\ \ 55\ \ 9\ .3$

The first of these triangles requires the positive sign in GAUSS's Equations, and the second requires the negative sign.

1. *Given* b, c, *and* A, *to find* a, B, and C. The general relations between the given and required parts are

$$\left.\begin{aligned} \cos a &= \cos c \cos b + \sin c \sin b \cos A \\ \sin a \cos B &= \sin c \cos b - \cos c \sin b \cos A \\ \sin a \sin B &= \sin B \sin A \end{aligned}\right\} \qquad (1)$$

and similar forms to the last two, with C and c interchanged with B and b.

The second members being computed, the numerical value and the sign of cos a will be determined from the first equation. From the second and third sin a and B are determined precisely as k and ϕ in the preceding section and are subject to the same ambiguity.* *The ambiguity will be removed, therefore, when the sign of either sin* a, *sin* B, *or cos* B *is given, and in like manner when the sign of either sin* C *or cos* C (*the other required parts*) *is given.*

The solution may be adapted for logarithmic computation, and the condition required for removing the ambiguity may be varied.

Let k and ϕ be determined by the conditions (9), taking $m = \sin b \cos A$, and $n = \cos b$, and adopting the first arbitrary condition; then these conditions together with equations (1) assume the following form:—

---

* For the second members of the 2d and 3d of (1) being computed, they will be fixed quantities like m and n in (9), and sin a occupies the place of k, and B that of ϕ in the same equations (9).

$$\left.\begin{aligned} k \sin \phi &= \sin b \cos \mathrm{A} \\ k \cos \phi &= \cos b \qquad (k \text{ positive}) \\ \cos a &= k \cos (c - \phi) \\ \sin a \cos \mathrm{B} &= k \sin (c - \phi) \\ \sin a \sin \mathrm{B} &= \sin b \sin \mathrm{A} \end{aligned}\right\} \qquad (2)$$

Or, eliminating k and adopting the third condition imposed on eqs. (9)

$$\left.\begin{aligned} \tan \phi &= \tan b \cos \mathrm{A} \qquad (\phi < 180^\circ) \\ \cos a &= \frac{\cos b}{\cos \phi} \cos (c - \phi) \\ \sin a \cos \mathrm{B} &= \frac{\cos b}{\cos \phi} \sin (c - \phi)^{*} \\ \sin a \sin \mathrm{B} &= \sin b \sin \mathrm{A} \end{aligned}\right\} \qquad (3)$$

If the quadrant in which a *is to be taken is given,* then

$$\left.\begin{aligned} \tan \phi &= \tan b \cos \mathrm{A} \qquad (\phi < 180^\circ) \\ \tan a \cos \mathrm{B} &= \tan (c - \phi)^{\dagger} \\ \tan a \sin \mathrm{B} &= \frac{\sin \phi \tan \mathrm{A}}{\cos (c - \phi)}^{\ddagger} \end{aligned}\right\} \qquad (4)$$

In (3) and (4) we may also limit ϕ to values numerically less than 90°, the sign of the tangent being determined according to the fifth arbitrary condition following (9).

If both a *and* b *are less than* 180°, as not unfrequently happens in the applications of this problem, let

$$m = \frac{k}{\sin b} \qquad n = \frac{\sin a}{k}$$

then m and n are both positive (k being positive) and the following form may be employed :—§

$$\left.\begin{aligned} m \sin \phi &= \cos \mathrm{A} \\ m \cos \phi &= \cot b \\ n \sin \mathrm{B} &= \sin \phi \tan \mathrm{A} \\ n \cos \mathrm{B} &= \sin (c - \phi) \\ \cot a &= \cot (c - \phi) \cos \mathrm{B}^{\|} \end{aligned}\right\} \qquad (5)$$

Check.—For the purpose of verification we may employ with (4) or (5), the formula $\sin a \sin \mathrm{B} = \sin b \sin \mathrm{A}$; and with any of the preceding solutions the following check :—

$$\frac{\sin (c - \phi)}{\sin \phi} = \frac{\sin a \cos \mathrm{B}}{\sin b \cos \mathrm{A}} = \frac{\tan \mathrm{A}}{\tan \mathrm{B}}^{\P}$$

---

[*] The value of k which appears in the 2d and 3d of (3) is obtained from the 2d of (2).

[†] Dividing the 3d by the 2d of (3).

[‡] Combining the 1st, 2d, and 4th of (3).

[§] Derived from (2).

[‖] Dividing the 3d by the 4th of (2).

[¶] From 1st, 4th, and 5th of (2).

2. *Given* b, c, *and* A, *to find* B *and* C. We employ Gauss's equations as follows:—

$$\left.\begin{array}{l}\cos\tfrac{1}{2}a\sin\tfrac{1}{2}(B+C)=\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(b-c)\\ \cos\tfrac{1}{2}a\cos\tfrac{1}{2}(B+C)=\sin\tfrac{1}{2}A\cos\tfrac{1}{2}(b+c)\\ \sin\tfrac{1}{2}a\sin\tfrac{1}{2}(B-C)=\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(b-c)\\ \sin\tfrac{1}{2}a\cos\tfrac{1}{2}(B-C)=\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(b-c)\end{array}\right\}\qquad(6)$$

The first two by eliminating cos ½ a determine ½ (B + C) when the sign of cos ½ a is known, and the second two determine ½ (B — C) when the sign of sin ½ a is known. *Hence, these equations present no ambiguity when the sign of sin* a *is given;* for sin ½ a is always positive, and cos ½ a has the same sign as sin a according to the formlua,

$$\sin a=2\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a$$

The equations (6) taken with the positive sign only may give values of B and C exceeding 360°, in which case the required solution will be found by diminishing such values by 360°.

3. *Given* B, C, *and* a *to find* A *and* b. The general relations between the given and required parts are

$$\left.\begin{array}{rl}\cos A= & -\cos C\cos B+\sin C\sin B\cos a\\ \sin A\cos b= & \sin C\cos B+\cos C\sin B\cos a\\ \sin A\sin b= & \sin B\sin a\end{array}\right\}\qquad(7)^{*}$$

which determine A and b without ambiguity, *when the sign of either* sin A, sin b, *or* cos b *is given.* In like manner the ambiguity is removed when the sign of either sin c or cos c is given.

Adapted for logarithms by the method already used, these equations become†

$$\left.\begin{array}{rl}k\sin\phi= & \sin B\cos a\\ k\cos\phi= & \cos B\qquad (k\ \text{positive})\\ \cos A= & -k\cos(C+\phi)\\ \sin A\cos b= & k\sin(C+\phi)\\ \sin A\sin b= & \sin B\sin a\end{array}\right\}\qquad(8)$$

Or,

$$\left.\begin{array}{rl}\multicolumn{2}{l}{\tan\phi=\tan B\cos a\ (\phi<180^\circ\ \text{always positive; or}\ \phi\ \text{less than}\ 90^\circ\ \text{with the sign of its tangent,})}\\ \cos A= & -\dfrac{\cos B}{\cot\phi}\cos(C+\phi)\\ \sin A\cos b= & \dfrac{\cos B}{\cos\phi}\sin(C+\phi)\\ \sin A\sin b= & \sin B\sin a\end{array}\right\}\qquad(9)\ddagger$$

When the quadrant in which A *is to be taken is known,*

---

* Derived from equations (6) p. 179 by polar triangles.

† Compare (2).

‡ Compare (3).

$$\left.\begin{aligned}
\tan\phi &= \tan B\cos a \quad (\phi < 180^\circ)\\
\tan A\cos b &= -\tan(C+\phi)\\
\tan A\sin b &= -\frac{\sin\phi\tan a}{\cos(C+\phi)}
\end{aligned}\right\}\qquad (10)^*$$

When A *and* B *are both less than* 180°;

$$\left.\begin{aligned}
m\sin\phi &= \cos a \qquad (m \text{ positive})\\
m\cos\phi &= \cot B\\
n\sin b &= \sin\phi\tan a \quad (n \text{ positive})\\
n\cos B &= \sin(C+\phi)\\
\cot A &= -\cot(C+\phi)\cos b
\end{aligned}\right\}\qquad (11)^\dagger$$

Check.—With (10) or (11) we may employ $\sin a\sin B = \sin b\sin A$, and with any of the solutions (8), (9), (10), (11), the check

$$\frac{\sin(C+\phi)}{\sin\phi} = \frac{\sin A\cos b}{\sin B\cos a} = \frac{\tan a}{\tan b}$$

4. *Given* B, C, *and* a, *to find* b *and* c. We employ GAUSS's equations arranged as follows:

$$\left.\begin{aligned}
\sin\tfrac{1}{2}A\sin\tfrac{1}{2}(b+c) &= \sin\tfrac{1}{2}a\cos\tfrac{1}{2}(B-C)\\
\sin\tfrac{1}{2}A\cos\tfrac{1}{2}(b+c) &= \cos\tfrac{1}{2}a\cos\tfrac{1}{2}(B+C)\\
\cos\tfrac{1}{2}A\sin\tfrac{1}{2}(b-c) &= \sin\tfrac{1}{2}a\sin\tfrac{1}{2}(B-C)\\
\cos\tfrac{1}{2}A\cos\tfrac{1}{2}(b-c) &= \cos\tfrac{1}{2}a\sin\tfrac{1}{2}(B+C)
\end{aligned}\right\}\qquad (12)$$

which present no ambiguity when the sign of $\cos\frac{1}{2}A$ is given; that is, *when the sign of* $\sin A$ *is given*, observing that $\sin\frac{1}{2}A$ is always positive, and $\cos\frac{1}{2}A$ has the same sign as $\sin A$.

As before, when these equations lead to values of b or c greater than 360°, the true values are to be found by subtracting 360°.

5. *Given* a, b, *and* A, *to find* B, C, *and* c. The general relations between the given and required parts, are

$$\left.\begin{aligned}
\sin a\sin B &= \sin b\sin A\\
-\cos C\cos A + \sin C\sin A\cos b &= \cos B\\
\sin C\cos A + \cos C\sin A\cos b &= \sin B\cos a\\
\cos c\cos b + \sin c\sin b\cos A &= \cos a\\
\sin c\cos b - \cos c\sin b\cos A &= \sin a\cos B
\end{aligned}\right\}\qquad (13)$$

The first equation determines B *when the sign of* $\cos B$ *is given;* and B being known, the remaining equations will fully determine C and c. Thus we find first

$$\sin B = \frac{\sin b\sin A}{\sin a}\qquad (14)$$

Then from the second and third of (13) if we put

$$\left.\begin{aligned}
k\sin\phi &= \cos A \qquad (k \text{ positive})\\
k\cos\phi &= \sin A\cos b\\
k\sin\phi' &= \cos B\\
k\cos\phi' &= \sin B\cos a \quad \text{we obtain}\\
C &= \phi+\phi'\S
\end{aligned}\right\}\qquad (15)^\ddagger$$

---

[*] Compare (4). † Compare (5). ‡ Compare with former check.

§ Because $\sin(C-\phi) = \sin\phi'$ and $\cos(C-\phi) = \cos\phi'$.

From the fourth and fifth, (1)

$$\left.\begin{aligned} k\sin\theta &= \sin b\cos A \quad (k \text{ positive})\\ k\cos\theta &= \cos b\\ k\sin\theta' &= \sin a\cos B\\ k\cos\theta' &= \cos a\\ c &= \theta+\theta' \end{aligned}\right\}\qquad(16)$$

In these solutions it may happen that $\phi+\phi'$, or $\theta+\theta'$ exceeds 360°, in which case $C=\phi+\phi'-360^\circ$, or $c=\theta+\theta'-360^\circ$.

Checks.—One of the following* may be employed when either C or c has alone been computed:—

$$\frac{\sin\phi}{\sin\phi'}=\frac{\cos A}{\cos B}\qquad\qquad\frac{\cos\phi}{\cos\phi'}=\frac{\tan a}{\tan b}$$

$$\frac{\sin\theta}{\sin\theta'}=\frac{\tan B}{\tan A}\qquad\qquad\frac{\cos\theta}{\cos\theta'}=\frac{\cos b}{\cos a}$$

When both C and c have been computed, the obvious check is

$$\frac{\sin C}{\sin c}=\frac{\sin A}{\sin a}$$

6. *Given* a, b, *and* A, *to find* C *and* c *without finding* B. Observing that k is positive in the preceding article we deduce the following forms and conditions, by eliminating B ;†

$$\left.\begin{aligned} k\sin\phi &= \cos A \qquad\qquad (k \text{ positive})\\ k\cos\phi &= \sin A\cos b\\ \cos\phi' &= \cos\phi\cot a\tan b \ (\phi' \text{ less}\\ &\text{than } 180^\circ\text{, with the same sign as } \cos B.)\ddagger\\ C &= \phi+\phi' \end{aligned}\right\}\qquad(17)$$

$$\left.\begin{aligned} k\sin\theta &= \sin b\cos A \qquad (k \text{ positive})\\ k\cos\theta &= \cos b\\ \cos\theta' &= \frac{\cos\theta\cos a}{\cos b}\,\S \qquad (\theta' \text{ less}\\ &\text{than } 180^\circ\text{, with the same sign as } \sin a\cos B), \end{aligned}\right\}\qquad(18)$$

Check.

$$\frac{\sin C}{\sin c}=\frac{\sin A}{\sin a}$$

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(1) The propriety of employing the same factor k in both (15) and (16) will be seen by comparing the values of k deduced from the two groups. We find in both cases $k^2=1-\sin^2 A\sin^2 b$.

* Deduced from (15) and (16).

† By substituting $\sin A\sin b$ for $\sin B$ in the 4th of (15), and for $\sin A$ its value $\frac{\cos\phi}{\sin a}$ from 2nd of 15.

‡ See 3d of (15.)

§ The value of $\cos\theta'$ is obtained by taking the value of k in the 2d of (16), and substituting it in the 4th.

In these solutions, when $\phi+\phi'$, and $+\theta+\theta'$ exceed 360°, we must take $C=\phi+\phi'-360^\circ$, $c=\theta+\theta'-360^\circ$; and when they are negative we must take $C=\phi+\phi'+360^\circ$, $c=\theta+\theta'+360^\circ$.

7. *Given* A, B, *and* a, *to find* b, c, *and* C. We find b by the formula

$$\sin b=\frac{\sin B\sin a}{\sin A}$$

which determines b *when the sign of* cos b* *is given.* The remainder of the solution is by (15) and (16).

8. *Given* A, B, *and* a, *to find* c *and* C *without finding* b. We may eliminate b from (15) and (16) in their present form, but the conditions for determining the auxiliary angles will not be so simple as in the following method. Let ϕ and ϕ' in (15) be exchanged for $\phi'-90^\circ$, and $\phi+90^\circ$ respectively; then after eliminating b, we find

$$\left.\begin{array}{l} k\sin\phi=-\sin B\cos a\ (k\text{ positive})\\ k\cos\phi=\cos B\\ \cos\phi'=-\dfrac{\cos\phi\cos A}{\cos B}\dagger\ (\phi'\text{ less}\\ \text{than }180^\circ\text{, with the sign of }\sin A\cos b.)\\ C=\phi+\phi' \end{array}\right\}\qquad(19)$$

In a similar manner from (15) we find

$$\left.\begin{array}{l} k\sin\theta=-\cos a\qquad(k\text{ positive})\\ k\cos\theta=\sin a\cos B\\ \cos\theta'=-\cos\theta\cot A\tan B\ddagger\ (\theta'\text{ less}\\ \text{than }180^\circ\text{ with the sign of }\cos b.)\\ c=\theta+\theta' \end{array}\right\}\qquad(20)$$

Check.

$$\frac{\sin C}{\sin c}=\frac{\sin A}{\sin a}$$

In these formulas, as before, when $\phi+\phi'$, and $\theta+\theta'$ exceed 360°, we take $C=\phi+\phi'-360^\circ$, $c=\theta+\theta'-360^\circ$; and when they are negative we take $C=\phi+\phi'+360^\circ$, $c=\theta+\theta'+360^\circ$.

9. *Given* a, b, *and* c, *to find* A, B, *or* C. The formula (see Art. 82)

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}\qquad(21)$$

determines A *when the sign of* sin A *is given;* or *when the sign of either* sin A, sin B, *or* sin C *is given;* when the sign of any one of these functions is known, those of the other two may be discovered by an inspection of the equation

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* Which determines the quadrant in which b is, the sign of the sin b only determining whether it is in the first two or last two.

† Compare the 3d of (18).

‡ Compare the 3d of (17).

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

The usual formulas for $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, $\tan \frac{1}{2} A$ (see Art. 84), derived from (21), may be employed, and the ambiguity removed, by the same conditions.

Check.—Compute two of the functions $\sin \frac{1}{2} A$, $\cos \frac{1}{2} A$, $\tan \frac{1}{2} A$; or one of them in connexion with (21).

10. *Given* A, B, *and* C, *to find* a, b, *or* c. The formula (see Art. 88)

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \qquad (22)$$

determines a *when the sign of* sin a *is given;* or *when the sign of either* sin a, sin b, *or* sin c *is given,* since when the sign of any one of these functions is known, those of the other two may be discovered by an inspection of the equation

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

the usual formulas for $\sin \frac{1}{2} a$, $\cos \frac{1}{2} a$, $\tan \frac{1}{2} a$, may be employed, and the ambiguity removed, by the same conditions.

Check.—Compute two of the functions $\sin \frac{1}{2} a$, $\cos \frac{1}{2} a$, $\tan \frac{1}{2} a$; or one of them in connexion with (22).

10. From the preceding sketch it appears that for the determinate solution of a spherical triangle generally considered, there are required *four* data; namely, the numerical values of three of the six parts composing the triangle, and the algebraic sign of one of the functions of a required part. To recapitulate, the triangle is fully determined by the following data:—

1. b, c, A; and the sign of either $\sin a$, $\sin B$, $\cos B$, $\sin C$, or $\cos C$.
2. B, C, a; and the sign of either $\sin A$, $\sin b$, $\cos b$, $\sin c$, or $\cos c$.
3. a, b, A; and the sign of $\cos B$.
4. A, B a; and the sign of $\cos b$.
5. a, b, c; and the sign of either $\sin A$, $\sin B$, or $\sin C$.
6. A, B, C; and the sign of either $\sin a$, $\sin b$, or $\sin c$.

---

11. Since π is the symbol which represents the circumference of a circle whose diameter is unity, or the semicircumference whose radius is unity, $\frac{1}{2}\pi$ will represent a quadrant of the latter or 90°. We may, therefore, for convenience, represent the supplement of an arc a by $\pi - a$, and its complement by $\frac{1}{2}\pi - a$.

12. A triangle, one side of which is π or 90°, is called a quadrantal triangle; such triangles may be resolved by Napier's rules for the circular parts, if the quadrantal side be neglected, and $\frac{1}{2}\pi - b$, $\frac{1}{2}\pi - c$, B, C, and $A - \frac{1}{2}\pi$ be taken for the circular parts.

For let $A'B'C'$ be the polar triangle. It will be right angled because $A = \pi - a = \frac{1}{2}\pi$. Applying Napier's rule to this triangle we obtain

$$\cos a' = \cos b' \cos c' = \cot B' \cot C' \qquad (1)$$

$$\sin b' = \sin B' \sin a = \cot C' \tan c' \qquad (2)$$

$$\cos B = \cos b \sin C = \cot a' \tan c \qquad (3)$$

These are the expressions for each of the parts of a right angled triangle in terms of two others, because expressions for c' and C' would be exactly like (2) and (3).

Substituting $\pi - A$, $\pi - a$, &c., for A', a', &c. in the above equations, they become

$$-\cos A = \cos B \cos c = \cot b \cot c$$
$$\sin B = \sin b \sin A = \cot c \tan C$$
$$-\cos b = -\cos B \sin c = \cot A \tan c$$

But these last equations are precisely what would be obtained by the application of Napier's rules, using the complements of b and c, and $A - \frac{1}{2}\pi$ as the circular parts.

13. Napier's rules may be deduced as follows:

The following formulas have been derived in the foregoing pages for oblique angled triangles.

Art. 82.

$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad (1)$$
$$\cos a \sin B \sin C = \cos A + \cos B \cos C \qquad (2)^*$$
$$\cos c \sin A \sin B = \cos C + \cos A \cos B \qquad (3)^*$$
$$\sin c \sin A = \sin a \sin C \qquad (4)^\dagger$$
$$\cot c \sin b = \cos b \cos A + \sin A \cot C \qquad (5)^\ddagger$$
$$\cot a \sin c = \cos c \cos B + \sin B \cot A \qquad (6)^\ddagger$$

Making in the above forms $A = 90^\circ$ they become

$$\cos a = \cos b \cos c \qquad (1)$$
$$\cos a = \cot B \cot C \qquad (2)$$
$$\cos C = \cos c \sin B \qquad (3)$$
$$\cos B = \cos b \sin C$$
$$\sin c = \sin a \sin C \qquad (4)$$
$$\sin b = \sin a \sin B$$
$$\sin b = \tan c \cot C \qquad (5)$$
$$\sin c = \tan b \cot B$$
$$\cos B = \cot a \tan c \qquad (6)$$
$$\cos C = \cot a \tan b$$

The above are but expressions of Napier's rules.

14. The case of solution treated at Art. 94, may be solved by Napier's Analogies. Thus if a, b and A be given, B may be calculated by the sin proportion, and C and c by the formulas

$$\sin \tfrac{1}{2}(a - b) : \sin \tfrac{1}{2}(a + b) :: \tan \tfrac{1}{2}(A - B) : \cot \tfrac{1}{2} C$$
$$\sin \tfrac{1}{2}(A - B) : \sin \tfrac{1}{2}(A + B) :: \tan \tfrac{1}{2}(a - b) : \tan \tfrac{1}{2} c$$

15. Napier's rules for the solution of right angled spherical triangles, though applicable to all cases, do not give results of that degree of accuracy which is some-

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* (2) and (3) are the same, and are derived from (1) by polar triangles.

† Sine proportion, Art. 81.

‡ Appendix II. See formula at top of p. 200.

times required, when the required part expressed by its sine is very small, or expressed by its cosine is very near 90°. The following formulas may in such cases be used.

I.

By formula (8), Art. 72, and p. 76, 2d note.

$$\left.\begin{aligned} 1 - \cos p &= 2 \sin^2 \tfrac{1}{2} p \\ 1 + \cos p &= 2 \cos^2 p \end{aligned}\right\} \quad (m)$$

whence

$$\frac{1 - \cos p}{1 + \cos p} = \tan^2 \tfrac{1}{2} p$$

But by Napier's rules, R being 1,

$$\cos a = \cot B \cot C$$

changing p into a, and substituting the value of $\cos a$, given by this last, we have

$$\tan^2 \tfrac{1}{2} a = \frac{1 - \cot B \cot C}{1 + \cot B \cot C} = \frac{\sin B \sin C - \cos B \cos C}{\sin B \sin C + \cos B \cos C}$$

or

$$\tan^2 \tfrac{1}{2} a = -\frac{\cos (B + C)}{\cos (B - C)}$$

which is a formula to be employed, when B and C are given and a required.

II.

With the same data to find b use the formula

$$\tan \tfrac{1}{2} b = \sqrt{\left\{\tan [\tfrac{1}{2} (B - C) + 45^\circ] \tan [\tfrac{1}{2} (B + C) - 45^\circ]\right\}}$$

derived from Napier's rules, which gives

$\cos b = \dfrac{\cos B}{\sin C}$, from the formulas preceding (1) and (4) of Art. 12, App. I. and from formula (42), Art. 15, App. I.

III.

The hypothenuse a and the side c being given to find the adjacent angle B, use the formula

$$\tan \tfrac{1}{2} B = \sqrt{\frac{\sin (a - c)}{\sin (a + c)}}$$

derived in a manner similar to that in case I.

IV.

Given a and b to find c; by Napier's rules

$$\cos c = \frac{\cos a}{\cos b}$$

add and subtract 1; by App. I., Art. 15, formula (52), and forms. (m) above

$$\tan \tfrac{1}{2} c = \tan \tfrac{1}{2} (a + b) \tan \tfrac{1}{2} (a - b)$$

V.

Finally, to obtain b when the opposite angle B and the hypothenuse a are given, we have, by Napier's rules,

$$\sin b = \sin a \sin B$$

whence, observing that $\frac{1 - \tan x}{1 + \tan x} = \tan (45^\circ - x)$, App. I., Art. 15, form. (43), and making $\tan x = \sin a \sin B$,

$$(\tan 45^\circ - \tfrac{1}{2} b) = \sqrt{\tan (45^\circ - x)}$$

16. The part of a spherical triangle determined by the proportion $\sin a : \sin b :: \sin A : \sin B$ admits of a double value, since two arcs answer to the same sine; it becomes necessary, therefore, for us to inquire under what circumstances both these values are admissible, and how we may know which to choose when but one solution exists.

Referring to the fundamental formula (Art. 82), we have

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}; \qquad (2)$$

in which expression we may remark that if $\cos b$ is numerically greater than either $\cos a$ or $\cos c$, the second member must take the sign of $\cos b$, consequently B and b must be of the same affection if $\sin b < \sin a$, or $\sin b < \sin c$, that is, *an angle must be of the same species as its opposite side, if the sine of this side is less than the sine of either of the other sides.*

But if $\cos b$ is numerically less than $\cos a$, then whether the right hand member be + or — will depend upon the magnitude of $\cos c$, or $\cos c$ will have two values corresponding to $+ \cos B$, and $- \cos B$; hence *an angle has two values, when the sine of its opposite side is greater than the sine of the other given side.*

In the proportion

$$\sin A : \sin B :: \sin a : \sin b$$

a being the required part, the nature of the arc b may be discussed, as in the preceding case.

By means of the polar triangles, we obtain from (2), in the same manner as at Art. 85, the formula

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C}$$

from which it follows, as in the foregoing case, that if $\cos B$ is numerically greater than $\cos A$, B and b will be of the same affection. If $\cos B$ is numerically less than $\cos A$, then both the values of b, given by the above proportion, will be admissible, for C may be determined so as to render $\cos b$ positive or negative. Hence *any side will be of the same affection as its opposite angle, if the sine of this angle be less than the sine of either of the other angles; and the affection of the side* b *will be indeterminate if the sine of its opposite angle* B *be greater than the sine of the other given angle* A.

17. In practice, the data for the solution of triangles are obtained by observation and measurement, and are liable to error from obvious and inevitable causes. It is true that from the great excellency of instruments, and the almost inconceivable accuracy of modern observation, these errors are extremely minute, yet in cases where precision is requisite, it becomes necessary to determine the effects which small errors in the data will produce upon the computed quantities, and to select the data and quæsita in such a manner that the given errors in the one shall entail the smallest possible on the other.

The principles of the Differential Calculus present an easy method for the purpose in question, and we shall here indicate the mode of proceeding, for the benefit of the student acquainted with that branch of mathematics.

Let us suppose that of the three data (for there are always three in the solution of a triangle), two have been obtained with sufficient accuracy, but the third x is liable to an error of a given amount, which we shall call h. Let u be the sought quantity. Two of the three data being considered constant, the sought quantity u may be considered as a function of the third x. The quantity x becoming $x + h$, let the quantity u become u', we have by Taylor's theorem,

$$u' - u = \frac{du}{dx} h + \frac{d^2u}{dx^2} h^2 + \frac{d^3u}{dx^3} h^3 + \&c.$$

$u' - u$ is the error sought, and as h is in practice very small, the higher powers of it may be neglected, and we may call

$$u' - u = \frac{du}{dx} h$$

hence the following rule:

Multply the given error by the differential coefficient of the sought quantity considered as a function of *the given quantity liable to error, and the product will be the error in the sought quantity.*

If two of the data be liable to given errors, the effect upon the sought quantity may be computed on similar principles, by considering the sought quantity as a function of the two data so liable to error, and differentiating it with respect to these two independent variables.

It is evident that the same method extends to the case where all the data are liable to given small errors. In this case the sought quantity is to be regarded as a fnuction of three independent variables, and its differential found as before.

EXAMPLE.

To determine the relation between the minute variations of the perpendicular side of a plane right angled triangle and the opposite angle, the remaining perpendicular side being considered constant.

Let c and c be the side and angle which are subject to variation, and b the constant side. Then (Art. 41),

$$c = b \tan \mathrm{c}$$

and (Dif. Cal.)

$$\frac{dc}{d\mathrm{c}} = b \sec^2 \mathrm{c}$$

which is the multiplier of the given small variation in c to obtain that in c.

The theory of maxima and minima as explained in the Calculus, will here admit of an important and easy application, viz., to find under what circumstances $u' - u$ will be least on the supposition of a given variation h in the variable datum, or in other words, under what circumstances the function $\frac{du}{dx}$ will be a minimum.

17. The effects of small errors may be obtained, but with less brevity and elegance, without the aid of the differential calculus.

The following are specimens of the mode of proceeding.

PROBLEM I.

In a right-angled triangle one of the oblique angles being given, to determine the variation of the opposite side, arising from a small variation in the hypothenuse.

Let c be the angle which does not change, c its opposite side, and a the hypothenuse; then δc denoting the variation of c, and δa of a

$$\sin c = \sin \mathrm{c} \sin a$$
$$\sin (c + \delta c) = \sin \mathrm{c} \sin (a + \delta a)$$

$\therefore$ by subtraction,

$$\sin (c + \delta c) - \sin c = \sin \mathrm{c} \left\{ \sin (a + \delta a) - \sin a \right\};$$

that is (Art. 74, formula 4),

$$2 \cos (c + \tfrac{1}{2} \delta c) \sin \tfrac{1}{2} \delta c = 2 \sin \mathrm{c} \cos (a + \tfrac{1}{2} \delta a) \sin \tfrac{1}{2} \delta a$$

$$\therefore \sin \tfrac{1}{2} \delta c = \frac{\sin \mathrm{c} \cos (a + \frac{1}{2} \delta a) \sin \frac{1}{2} \delta a}{\cos (c + \frac{1}{2} \delta c}$$

and if δa, δc be very small,

$$\delta c = \frac{\sin \mathrm{c} \cos a}{\cos c} \delta a;$$

or, substituting for $\sin \mathrm{c}$ its value from the first equation,

$$\delta c = \frac{\sin c}{\cos c} \cdot \frac{\cos a}{\sin a} \delta a = \tan c \cot a \, \delta a;$$

which variation will be the least possible when $\cot a$ is least, or when $a = 90°$. If we restore the $\frac{1}{2} \delta a$ which has been neglected, and write the above result thus:

$$\delta c = \tan c \cot (a + \tfrac{1}{2} \delta a) \, \delta a;$$

then, in the case of $a = 90°$, the expression becomes

$$\delta c = - \tan c \tan \tfrac{1}{2} \delta a \cdot \delta a;$$

or, considering the very small arc $\frac{1}{2} \delta a$ to be equal to its tangent, we have in the case supposed

$$\delta c = - \tfrac{1}{2} \tan c \, (\delta a)^2.$$

PROBLEM II.

In an oblique-angled spherical triangle given two sides to determine the variation produced in the third side by a small variation of the opposite angle.

Let a, b be the two given sides, C the included angle, and c the opposite side. Then

$$\cos c = \cos a \cos b + \sin a \sin b \cos \mathrm{C}$$
$$\cos (c + \delta c) = \cos a \cos b + \sin a \sin b \cos (\mathrm{C} + \delta \mathrm{C})\,;$$

∴ by subtraction,

$$\cos (c + \delta c) - \cos c = \sin a \sin b \left\{\cos (\mathrm{C} + \delta \mathrm{C}) - \cos \mathrm{C}\right\};$$

that is,

$$2 \sin (c + \delta c) \sin \tfrac{1}{2}\, \delta c = 2 \sin a \sin b \sin (\mathrm{C} + \tfrac{1}{2}\, \delta \mathrm{C}) \sin \tfrac{1}{2}\, \delta \mathrm{C}$$

Hence, if $\delta \mathrm{C}$ be very small,

$$\sin c\, \delta c = \sin a \sin b \sin \mathrm{C}\, \delta \mathrm{C}$$
$$\therefore\ \delta c = \frac{\sin a \sin b \sin \mathrm{C}}{\sin c} \delta \mathrm{C}$$
$$= \sin a \sin \mathrm{B} \delta \mathrm{C}\,;$$

and δc is therefore the least possible when $\sin \mathrm{C}$ is the least possible, that is, when $\mathrm{C} = 0$. To find the expression for δc, in this case, restore what has been rejected, and we shall have

$$\delta c = \frac{\sin a \sin b \sin\ (\mathrm{C} + \tfrac{1}{2}\, \delta \mathrm{C})}{\sin c} \delta \mathrm{C}\,;$$

which, when $\mathrm{C} = 0$, and $\tfrac{1}{2}\, \delta \mathrm{C}$ very small, becomes

$$\delta c = \frac{\sin a \sin b}{2 \sin c} (\delta \mathrm{C})^2$$

PROBLEM III.

In an oblique angled spherical triangle given two sides and the included angle, to find the variation in one of the opposite angles corresponding to a small variation in the included angle.

Let a, b be the given side, C the included angle, to find what influence a small variation in the value of C will have on A opposite a. Substitute the expression for $\cos c$ above, in the corresponding expression for $\cos a$, and $1 - \sin^2 b$ for $\cos^2 b$, there results

$$\cos \mathrm{A} \sin c = \cos a \sin b - \sin a \cos b \cos \mathrm{C}\,;$$
$$\therefore\ \cos \mathrm{A} \frac{\sin c}{\sin a} = \cot a \sin b - \cos b \cos \mathrm{C}$$

But

$$\frac{\sin c}{\sin a} = \frac{\sin \mathrm{C}}{\sin \mathrm{A}}$$

Then, by substitution,

$$\cot A \sin C = \cot a \sin b - \cos b \cos C$$

$$\cot (A + \delta A) \sin (C + \delta C) = \cot a \sin b - \cos b\,(C + \delta C)\,;$$

and, by subtraction,

$$\cot (A + \delta A) \sin (C + \delta C) - \cot A \sin C = \cos b \left\{\cos C - \cos (C + \delta C\right\} \qquad (1)$$

The first member of this equation is equal to

$$\cot (A + \delta A)\left\{\sin (C + \delta C) - \sin C\right\} + \sin C\left\{\cot (C + \delta A) - \cot A\right\}\,;$$

and the quantities within the brackets are respectively equal to

$$2 \cos (C + \tfrac{1}{2}\,\delta C) \sin \tfrac{1}{2}\,\delta C \text{ and } \frac{-\sin \delta A}{\sin A \sin (A + \delta A)}.$$

The second member of (1) is equal to

$$\cos b \cdot 2 \sin (C + \tfrac{1}{2}\,\delta C) \sin \tfrac{1}{2}\,\delta C$$

$$\therefore 2 \cot (A + \delta A) \cos (C + \tfrac{1}{2}\,\delta C) \sin \tfrac{1}{2}\,\delta C - \frac{\sin C \sin \delta A}{\sin A \sin (A + \delta A)} = 2 \cos b \sin (C + \tfrac{1}{2}\,\delta C) \sin \tfrac{1}{2}\,\delta C$$

and consequently when δC and δA are very small,

$$\cot A \cos C\,\delta C - \frac{\sin C}{\sin^2 A}\,\delta A = \cos b \sin C\,\delta C$$

$$\therefore \delta A = \frac{\sin^2 A}{\sin C}\,(\cot A \cos C - \cos b \sin C)\,\delta C$$

TABLES

TABLE XXIV.

Of Natural Sines.

Prop. parts		0°		1°		2°		3°		4°			Prop. parts
29	M	N. sine.	N. cos.	N. sine.	N. cos.	N. sine.	N. cos.	N. sine	N. cos.	N. sine.	N. cos.		2
0	0	00000	100000	01745	99985	03490	99939	05234	99863	06976	99756	60	2
0	1	00029	100000	01774	99984	03519	99938	05263	99861	07005	99754	59	2
1	2	00058	100000	01803	99984	03548	99937	05292	99860	07034	99752	58	2
1	3	00087	100000	01832	99983	03577	99936	05321	99858	07063	99750	57	2
2	4	00116	100000	01862	99983	03606	99935	05350	99857	07092	99748	56	2
2	5	00145	100000	01891	99982	03635	99934	05379	99855	07121	99746	55	2
3	6	00175	100000	01920	99982	03664	99933	05408	99854	07150	99744	54	2
3	7	00204	100000	01949	99981	03693	99932	05437	99852	07179	99742	53	2
4	8	00233	100000	01978	99980	03723	99931	05466	99851	07208	99740	52	2
4	9	00262	100000	02007	99980	03752	99930	05495	99849	07237	99738	51	2
5	10	00291	100000	02036	99979	03781	99929	05524	99847	07266	99736	50	2
5	11	00320	99999	02065	99979	03810	99927	05553	99846	07295	99734	49	2
6	12	00349	99999	02094	99978	03839	99926	05582	99844	07324	99731	48	2
6	13	00378	99999	02123	99977	03868	99925	05611	99842	07353	99729	47	2
7	14	00407	99999	02152	99977	03897	99924	05640	99841	07382	99727	46	2
7	15	00436	99999	02181	99976	03926	99923	05669	99839	07411	99725	45	2
8	16	00465	99999	02211	99976	03955	99922	05698	99838	07440	99723	44	1
8	17	00495	99999	02240	99975	03984	99921	05727	99836	07469	99721	43	1
9	18	00524	99999	02269	99974	04013	99919	05756	99834	07498	99719	42	1
9	19	00553	99998	02298	99974	04042	99918	05785	99833	07527	99716	41	1
10	20	00582	99998	02327	99973	04071	99917	05814	99831	07556	99714	40	1
10	21	00611	99998	02356	99972	04100	99916	05844	99829	07585	99712	39	1
11	22	00640	99998	02385	99972	04129	99915	05873	99827	07614	99710	38	1
11	23	00669	99998	02414	99971	04159	99913	05902	99826	07643	99708	37	1
12	24	00698	99998	02443	99970	04188	99912	05931	99824	07672	99705	36	1
12	25	00727	99997	02472	99969	04217	99911	05960	99822	07701	99703	35	1
13	26	00756	99997	02501	99969	04246	99910	05989	99821	07730	99701	34	1
13	27	00785	99997	02530	99968	04275	99909	06018	99819	07759	99699	33	1
14	28	00814	99997	02560	99967	04304	99907	06047	99817	07788	99696	32	1
14	29	00844	99996	02589	99966	04333	99906	06076	99815	07817	99694	31	1
15	30	00873	99996	02618	99966	04362	99905	06105	99813	07846	99692	30	1
15	31	00902	99996	02647	99965	04391	99904	06134	99812	07875	99689	29	1
15	32	00931	99996	02676	99964	04420	99902	06163	99810	07904	99687	28	1
16	33	00960	99995	02705	99963	04449	99901	06192	99808	07933	99685	27	1
16	34	00989	99995	02734	99963	04478	99900	06221	99806	07962	99683	26	1
17	35	01018	99995	02763	99962	04507	99898	06250	99804	07991	99680	25	1
17	36	01047	99995	02792	99961	04536	99897	06279	99803	08020	99678	24	1
18	37	01076	99994	02821	99960	04565	99896	06308	99801	08049	99676	23	1
18	38	01105	99994	02850	99959	04594	99894	06337	99799	08078	99673	22	1
19	39	01134	99994	02879	99959	04623	99893	06366	99797	08107	99671	21	1
19	40	01164	99993	02908	99958	04653	99892	06395	99795	08136	99668	20	1
20	41	01193	99993	02938	99957	04682	99890	06424	99793	08165	99666	19	1
20	42	01222	99993	02967	99956	04711	99889	06453	99792	08194	99664	18	1
21	43	01251	99992	02996	99955	04740	99888	06482	99790	08223	99661	17	1
21	44	01280	99992	03025	99954	04769	99886	06511	99788	08252	99659	16	1
22	45	01309	99991	03054	99953	04798	99885	06540	99786	08281	99657	15	1
22	46	01338	99991	03083	99952	04827	99883	06569	99784	08310	99654	14	0
23	47	01367	99991	03112	99952	04856	99882	06598	99782	08339	99652	13	0
23	48	01396	99990	03141	99951	04885	99881	06627	99780	08368	99649	12	0
24	49	01425	99990	03170	99950	04914	99879	06656	99778	08397	99647	11	0
24	50	01454	99989	03199	99949	04943	99878	06685	99776	08426	99644	10	0
25	51	01483	99989	03228	99948	04972	99876	06714	99774	08455	99642	9	0
25	52	01513	99989	03257	99947	05001	99875	06743	99772	08484	99639	8	0
26	53	01542	99988	03286	99946	05030	99873	06773	99770	08513	99637	7	0
26	54	01571	99988	03316	99945	05059	99872	06802	99768	08542	99635	6	0
27	55	01600	99987	03345	99944	05088	99870	06831	99766	08571	99632	5	0
27	56	01629	99987	03374	99943	05117	99869	06860	99764	08600	99630	4	0
28	57	01658	99986	03403	99942	05146	99867	06889	99762	08629	99627	3	0
28	58	01687	99986	03432	99941	05175	99866	06918	99760	08658	99625	2	0
29	59	01716	99985	03461	99940	05205	99864	06947	99758	08687	99622	1	0
29	60	01745	99985	03490	99939	05234	99863	06976	99756	08716	99619	0	0
		N. cos.	N. sine.	N. cos.	N. sine.	N. cos.	N. sine.	N. cos.	N. sine.	N. cos.	N. sine.	M	
		89°		88°		87°		86°		85°			

PART III.

WHEN a ship sails from any known place, and a correct account is kept of her various directions, and rates of sailing, her situation at any time may be determined by the rules of Plane Trigonometry. The processes employed for this purpose constitute what is called *Navigation.*

But, owing to the imperfection of the instruments with which a ship's course and the distance sailed are observed, it would be unsafe, after a long passage, to compute the place of the ship from the *dead reckoning*, as the observed direction and distance are called. In such cases recourse must be had to astronomical observations, from which the place of the ship or its *latitude* and longitude are computed by the rules of Spherical Trigonometry. The problem then becomes one of *Nautical Astronomy.* We shall treat successively of each of these important branches.

NAVIGATION.

DEFINITIONS.

96. 1. For the purposes of Navigation the earth may be considered as spherical. It revolves about one of its diameters, called its *axis*, in twenty-four hours. This rotation is from west to east, causing the heavenly bodies to have an apparent motion from east to west.

2. The great circle, whose poles are the extremities of the axis, is called the *equator*. The poles of the equator are called also the *poles* of the earth; the one being the *north pole*, and the other the *south pole.*

3. Great circles passing through the poles are called *meridians.* Through every place on the surface of the earth such a great circle may be drawn, and will be the meridian of the place. The meridian from which the meridians of other places are estimated is called a *first meridian.*

The English have fixed upon the meridian of Greenwich Observatory for the first meridian, which has also been adopted in this country.

4. The *longitude* of any place is the arc of the equator, intercepted between the meridian of that place and the first meridian; the longitude, therefore, is the measure of the angle between the planes of the two meridians. The longitude is east or west, according as the place is situated east or west of the first meridian.

5. The *difference of longitude* between two places is the arc of the equator intercepted between the meridians of those places, or the measure of the angle which their planes include; hence, when the longitudes of the places are of the *same* denomination, that is, either both east or both west, the difference is found by *subtracting* the one from the other; but when they are of *contrary* denominations the difference is found by *adding* the one to the other.

6. The *latitude* of a place is its distance north or south of the equator, measured on the meridian of the place. Latitude cannot exceed 90°.

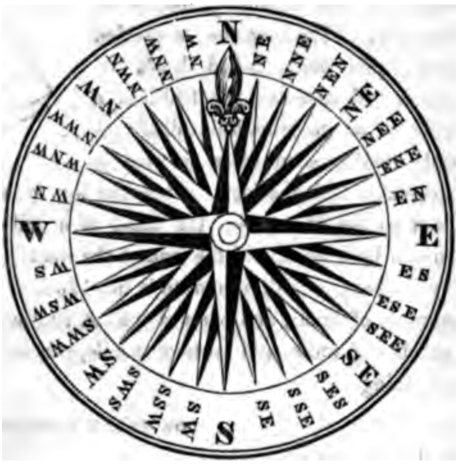
7. Small circles parallel to the equator, are called *parallels* of latitude. The arc of a meridian, intercepted between two such parallels, drawn through any two places, is the *difference of latitude* of those places; when the latitudes are of the same name, *i. e.*, both N. or both S., the difference of latitude is found by subtraction, but when not, the difference of latitude is found by addition.

8. The *horizon* of any place is an imaginary plane, touching the surface of the earth at that place, and extending to the heavens; such a plane is called the *sensible horizon*, and one parallel to it, but passing through the centre of the earth, the *rational horizon* of the place. The line of intersection of the plane of the horizon, and the plane of the meridian of a place, is called a north and south line; the horizontal line through the same point, and perpendicular to this, is called the east and west line. Besides the North, South, East, and West points, called cardinal points, thus determined on the boundary of the horizon, there are numerous subdivisions corresponding to the divisions in the circle on the next page.

9. The *course* of a ship is the angle which her track makes with the meridians; if this angle continued the same, and the meridians were all parallel, the path of the ship would be a straight line; but as the meridians bend towards the poles, the direction of her path is continually changing, and she moves in a curve, called the *rhumb line* or *loxodromic curve*. The instrument employed on ship-board to show the course of the ship is called the *mariner's compass*.

10. *The Mariner's Compass* consists of a circular card, whose circumference is divided into thirty-two equal parts, called *points*, and each

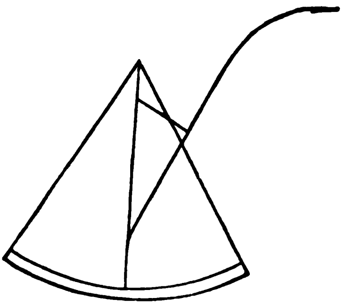
of these is subdivided into four equal parts, called quarter points; across this card, in the direction of a diameter, and fastened to the card, so that they move together, is fixed a slender bar of magnetized steel, called *the needle;* the extremities of which point to two diametrically opposite divisions of the card. These opposite divisions are marked N. and S., corresponding to the *north* and *south* poles, or ends, of the magnetized bar. The diameter E.W., at right angles to the diameter N.S., points out the east and west points.



One point from the north towards the east, is marked N.E., and called *north by east;* two points, N.N.E., and called *north north-east;* three points, *north-east by north;* and so on. Each quadrant contains eight points, so that a point is $90^\circ + 8 = 11^\circ\ 15'$. (See Table of Rhumbs, Table XXVIII., at the end.)

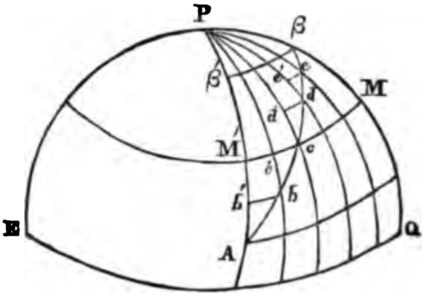
The card thus furnished being now suspended horizontally, so as to move freely, and allow the needle attached to it, to settle itself, will point out the four cardinal points of the horizon, as also the several intermediate points, provided only that it is the property of the magnetic needle to point due north and south. Such, however, is not strictly the case, as is found by comparison with astronomical observations. The card rests at its centre, on a pivot placed in the vertical plane, cutting the ship from stem to stern, and is held stationary in space by the magnetic forces of the earth, whilst the ship turns under it in changing her course, so that that point of the compass which is directed to the ship's head shows the *ship's course*, which must be corrected for the slight variation of the compass from the meridian, a variation which is different in different parts of the earth; the method of determining it will be hereafter given.

11. A ship's *rate* of sailing is determined by means of an instrument called the *log*, and an attached line, called the *log-line.* The log is a piece of wood in the form of the sector of a circle, the rim of which is so loaded with lead, that when *heaved* into the sea it assumes a vertical position, having its centre barely above the water. The log-line is so



attached as to keep the face of the log towards the ship, that it may offer the greater resistance to being dragged after the ship by the log-line, as it unwinds from a reel on board, by the advancing motion of the ship. The log-line is divided into equal parts, called *knots*, of which each measures the 120th of a nautical or geographical mile.* A half minute sand glass is used in connexion with the log. When the log is heaved, the instant the first knot on the line passes the hand of a sailor, the half minute glass is turned by a word, and the instant the sand is run out, the line is caught by a word; as half a minute is the 120th of an hour, it follows that the number of knots, and parts of a knot, run in half a minute expresses the number of miles, and parts of a mile, run in an hour, at the same rate of sailing.

ON PLANE SAILING.



97. Let the annexed diagram represent a portion of the earth's surface, P the pole, and E Q the equator. Let A β be a rhumb line, or path described by a ship in sailing on a single course from A to β Let the rhumb line be divided into portions A*b*, *bc*, *cd*, &c., so small that each may differ insensibly from a straight line, and draw meridians through these several divisions, as also the parallels of latitude bb', cc', dd', &c.; a series of triangles will thus be described on the surface of the globe, but so small that each may be considered as a plane triangle. These triangles are all similar, for the angles at b', c', d', &c., are right-angles, and the ship's path cuts all the meridians at equal angles; hence (Theorem 63 Geom.),

$$Ab : Ab' :: bc : bc' :: cd : cd', \&c.$$

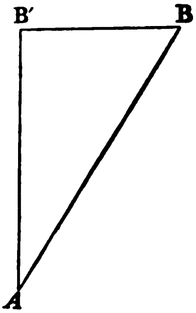
---

* The geographical mile is one minute of the earth's circumference. Taking the diameter at 7916 English miles the geographical mile will be about 6079 feet.

therefore, since by the theory of proportion the sum of the antecedents is to the sum of the consequents as any one antecedent is to its consequent,

$$\text{A}b : \text{A}b' :: \text{A}b + bc + cd + \text{\&c.}, : \text{A}b' + bc' + cd' + \text{\&c.}$$

But $\text{A}b + bc + cd +$ &c., is the whole distance sailed, and $\text{A}b' + bc' + cd' +$ &c. $= \text{A}\beta'$, is the difference of latitude between A and β; consequently, if a right angled triangle ABB′, similar to the small triangle Abb' be constructed, that is, one in which the angle A is equal to the course, and the hypothenuse A B is equal to the distance sailed, the side A B′ will represent the difference of latitude. Moreover, the other side BB′, or that opposite to the course, will represent the sum $b'b + c'c + d'd +$ &c. of all the minute *departures* which the ship makes from the successive meridians which it crosses; for as the triangle ABB′, in this last diagram, is similar to the small triangle Abb', in the former we have



$$\text{A}b : bb' :: \text{AB} : \text{BB}' \qquad (1)$$

but in the first figure we have

$$\text{A}b : bb' :: bc : cc' :: cd : dd', \text{\&c.}$$

$$\therefore \text{A}b : bb' :: \text{A}b + bc + cd + \text{\&c.} : bb' + cc' + dd' + \text{\&c.} \qquad (2)$$

consequently, since the three first terms of (1) are respectively equal to those of (2), the fourth term BB′, of (1), must be equal to the fourth term, $bb' + cc' + dd' +$ of (2), &c. This last quantity is called the *departure* of the ship in sailing from A to β.* It follows, therefore, that the *distance sailed*, the *difference of latitude* made, and the *departure*, are correctly represented by the *hypothenuse* and *sides* of a right angled plane triangle, in which the angle opposite the departure is the *course*, so that when any two of these four things are given, the other two may be found simply by the resolution of a right angled plane triangle; so far, therefore, as these particulars are concerned, the results are the same as if the ship were sailing on a plane surface, the meridians being parallel straight lines, and the parallels of latitude cutting them at right angles; and hence that part of Navigation in which only distance sailed, departure, difference of latitude, and course are considered, is called *Plane Sailing.*

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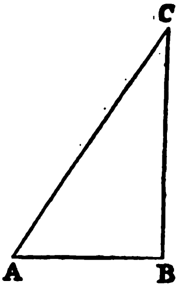
* The departure is not to be confounded with $\beta\beta'$ in the first diagram. It is greater than this, because the small departures bb', cc', &c., whose sum is the whole departure, lap over each other.

The two of the four elements which enter into problems of plane sailing usually given are course and distance, being found from observation.

EXAMPLES.

1. A ship from latitude 47° 30′ N. has sailed S.W. by S. 98 miles. At what latitude has she arrived, and what departure has she made?

Let C be the place sailed from, CB the meridian, the angle C = 3 points = 33° 45′, see Table of Rhumbs, and CA = 98 miles, the distance sailed; then CB will be the difference of latitude, and BA the departure. Then by the formulas for the solution of right angled triangles (forms (4) and (5) Art. 64)



Dist.	98	log.	1•99123	log.	1•99123
Course	33° 45′	log. cos	9•91985	log. sin	9•74474
Diff. of lat.	81•48	log.	1•91108		
Dep.	54•45			log.	1•73597

Latitude left 47° 30′ N.

Diff. of lat. = 81•48* minutes = 1 22 S. Dep. = 52•45 miles W.

Latitude in 46 9 N.

2. A ship sails for 24 hours on a direct course, from lat. 38° 32′ N., till she arrives at lat. 36° 56′ N.; the course is between S. and E., and the rate 5½ miles an hour. Required the course, distance and departure.

Lat. left 38° 32′ N. 24 × 5½ = 132 miles, the distance.

Lat. in 36 56 N.

Diff. 1 36 = 96 miles.

Dist.	132	log.	2•12057	log.	2•12057
Diff. lat.	96	log.	1•98227		
Course	43° 20′	log. cos	9•86170	log. sin	9•83648
Dep.	90•58			log.	1•95705

Hence the course is S. 43° 20′ E., and the departure 90•58 miles E.

3. A ship sails from lat 3° 52′ S. to lat. 4° 30′ N., the course being N.W. by W. ½ W.; required the distance and departure.

Distance, 1065 miles; Departure 939•2 miles W.

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* See last note but one. Miles are converted into degrees, &c., by dividing by 60.

4. Two ports lie under the same meridian, one in latitude 52° 30′ N. and the other in latitude 47° 10′ N. A ship from the southernmost sails due east, at the rate of 9 miles an hour, and two days after meets a sloop which had sailed from the northernmost port; required the sloop's direct course and distance run.

Course S. 53° 28′ E., or S.E. ¾ E.; distance run 537·6 miles.

5. If a ship from lat. 48° 27′ S. sail S.W. by W. 7 miles an hour, in what time will she arrive at the parallel of 50° S?

In 23·914 hours.

6. If after a ship has sailed from lat. 40° 21′ N. to lat. 46° 18′ N., she be found 216 miles to the eastward of the port left; required her course and distance sailed. Course N. 31° 11′ E., distance 417·3 miles.

TRAVERSE SAILING

98. Is where a ship, in going from one place to another, sails on different courses; the determination of the single course and distance from the one place to the other is called *working* or *compounding* the traverse. The method of proceeding is to rule seven columns (see next page), the first to contain the courses, the second the number of points and quarter points in these courses which may be found from the table of Rhumbs (Tab. XXVIII.), the third the distances sailed on these courses, the fourth and fifth the differences of latitude in two columns entitled N. and S., in the former of which the dif. of lat. for the N. courses is entered, and in the latter of which the dif. of lat. for the S. courses; the sixth and seventh contain the departures, the former those of the E. courses, the latter those of the W. courses, sometimes called *eastings* and *westings*.

When these several particulars are all inserted, the columns are added up, and the difference of the sums of the N. and S. columns will be the whole difference of the latitude which the ship has made, and the difference of the sums of the E. and W. columns will be her whole departure.

The columns appropriated to the difference of latitude and departures are usually filled up from a table (Table I.), already computed to every quarter point of the compass, and to all distances from one mile up to 300; so that by entering this table with any given course and distance, the proper difference of latitude and departure is found by inspection.

The course is found at the top or bottom of the page, and the distance in the first column, the dif. of lat. in the second column, and the departure in the third, if the course be found at top; but if at the bottom, the lat. and dep. columns are interchanged, as may be seen by the entitling in the bottom of the columns. If the distance sailed be more than 300 miles, it will exceed the limit of the table; but the difference of latitude and departure may still be determined from it by this simple operation: divide the given distance by any number that will give a quotient not exceeding 300; enter the table with this quotient, and multiply the corresponding diff. of lat. and dep. by the assumed divisor, and there will result the diff. of lat. and dep. due to the proposed distance. If there be a remainder add the diff. of lat. and dep. corresponding to this. Or take any two numbers whose sum is equal to the given distance; the sum of their differences of lat. and dep. will be the lat and dep. of the given distance. These rules depend upon the principle that for the same course the differences of latitude and departure are proportional to the distance run, which will be evident if we recollect that dist., diff. of lat. and dep. form a right angled triangle, and that two right angled triangles are similar when an acute angle of one is equal to an acute angle of the other.

EXAMPLE.

1. A ship sails from lat. 24° 32′ N., and runs the following courses and distances, viz.

1st, S.W. by W., distance 45 miles; 2d, E. S. E., distance 50 miles; 3d, S.W., distance 30 miles; 4th, S.E. by E., distance 60 miles; 5th, S.W. by S. ¼ W., distance 63 miles: required her present latitude, with the direct course and distance from the place left, to the place arrived at.

TRAVERSE TABLE.

COURSES.	PTS.	DIST.	DIFFERENCE OF LAT.		DEPARTURE.	
			N.	S.	E.	W.
S. W. by W.	5	45		25·0		37·4
E. S. E.	6	50		19·1	46·2	
S. W.	4	30		21·2		21·2
S. E. by E.	5	60		33·3	49·9	
S. W. by S. ¼ W.	3¼	63		50·6		37·5
				149.2	96·1	96·1

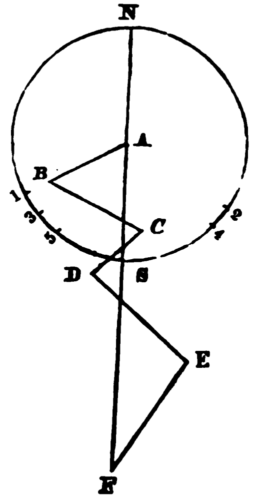
It appears from the results of this table that the difference of latitude made by the ship during the traverse is 149·2 S. =2° 29′ S.

Lat. left	24° 32′	N.
Diff. lat.	2 29	S.
Lat. in	22 3	N.

It appears also that the departures east are equal to the departures west, so that the ship has returned to the meridian she sailed from, consequently the direct course from the place left to that come to is due south, and the distance is equal to the difference of latitude, which is 149·2 miles.

There is another mode of finding the direct course and distance, much practised by seamen, viz., by construction. For this purpose the *mariner's scale* is employed, which is a two foot flat rule exhibiting several scales on each side, by help of which and a pair of compasses the usual problems in sailing may be all solved. One of these scales, which is called a scale of rhumbs, is a scale of chords, to every point and quarter point of the compass; and another is a more enlarged scale of chords to every degree. Both these scales are constructed for a circle with the same common radius, so that the chords on the scale of rhumbs belong to that circle whose radius equals the chord of 60° on the scale of chords; and the method of laying down a traverse from these scales, and a scale of equal parts, and of thence measuring the equivalent single course, and distance made good, will be at once understood from the example.

Construction of the traverse for the last example is as follows:



With the chord of 60°, taken from the line of chords on the mariner's scale, describe the horizon circle, and draw the north and south line N. S. From the line of rhumbs take the chords of the several courses, and as these are all southerly they must be laid off from the south point S, those which are westerly to the left, and those which are easterly to the right, their extremities being marked 1, 2, 3, &c., in the order of the courses. This done, lay off from any scale of equal parts, and in the direction of A1, the distance AB sailed on the first course; then in the direction parallel to A2, the distance BC sailed on the second course; in the direction parallel to A3, the distance CD on the third course; in the direction parallel to

A4, the distance DE on the fourth course; and, lastly, in the direction parallel to A5, the distance EF on the fifth course; then F will represent the place of the ship at the end of the traverse; FA, being applied to the scale of equal parts, will show the distance made good, and the chord of the arc included between this distance, and the meridian, being applied to the line of rhumbs, will show the direct course. In the present case the intercepted arc will be 0, showing that F is on the meridian of A.

2. A ship from Cape Clear, in lat 51° 25′ N., sails 1st, S.S.E. ¼ E., 16 miles; 2d, E.S.E., 23 miles; 3d, S.W. by W. ½ W., 36 miles; 4th, W. ¾ N., 12 miles; 5th, S.E. by E. ¼ E., 41 miles: required the distance made good, the direct course, and the latitude reached?

TRAVERSE TABLE.

COURSES.	PTS.	DIST.	DIFFERENCE OF LAT.		DEPARTURE.	
			N.	S.	E.	W.
S.S.E. ¼ E.	2¼	16		14·5	6·7	
E.S.E.	6	23		8·8	21·3	
S.W. by W. ½ W.	5½	36		17·0		31·8
W. ¾ N.	7¼	12	1·8			11·9
S.E. by E. ¼ E.	5¼	41		21·1	35·2	
			1·8	61·4	63·3	43·3
				1·8	43·7	
				59·6	19·6	

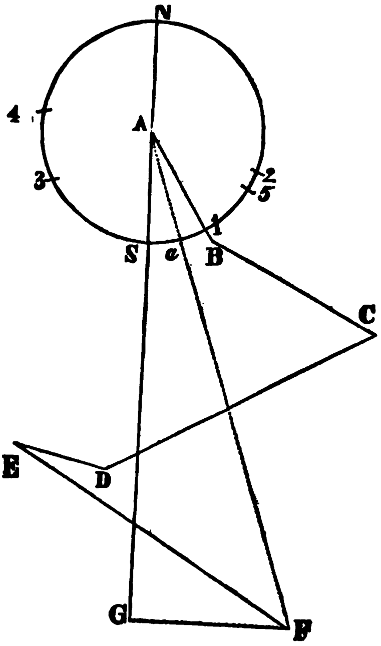
Lat. left	51° 25′ N.
Diff. lat 59·6 miles	1 0 S.
Lat. in	50 25 N.

Then by means of the whole dif. of lat. and dep., which are the two perpendicular sides of a right angled triangle, one of the acute angles and hypothenuse, or the direct course and distance, may be computed as follows:

diff. lat. 59·6 log.	1·77525		
departure 19·6 log.	1·29226	log.	1·29226
course 18° 12′ log. tan	9·51701	log. sin	9·49462
			1·79764
distance 62·74			

therefore, as the difference of latitude is south, and the departure east, the direct course is S. 18° 12′ E., and the distance made good 62·74 miles.

To construct this traverse, describe, as before, the horizon circle, with a radius equal to the chord of 60°, and taking from the line of rhumbs the chord of the first course, 2¼ points, apply it from S to 1, to the right of S N, as this course is south-easterly; apply, in like manner, the chord of the second course, six points from S to 2, also to the right of the meridian line; apply the chord of the third course, 5½ points from S to 3, to the left of the meridian, the chord of the fourth course, 7¼ from N. to 4, to the left of N S, this course being north-westerly, and, lastly, apply the chord of the fifth course, 5¼ points, from S to 5, to the right of S N. In the direction A1, lay off the distance AB = 16 miles from a scale of equal parts; in the direction parallel to A2, lay off the distance BC = 23 miles; in the direction parallel to A3, lay off CD = 36; in the direction parallel to A4 lay off DE = 12 miles; and, lastly, in the direction parallel to A5, lay off EF = 41; then F will be the place of the ship at the end of the traverse; consequently, AF will be the distance made good, and the angle FAS the direct course; applying, therefore, the distance AF to the scale of equal parts, we shall find it reach from 0 to 62¾; and applying the distance S*a* to the line of chords, we shall find it reach from 0 to 18°, or by the scale of rhumbs it will be found to measure one point and a half.



3. A ship from lat. 28° 32′ N., has run the following courses, viz., 1st, N.W. by N., 20 miles; 2d, S.W., 40 miles; 3d, N.E. by E., 60 miles; 4th, S.E. 55 miles; 5th, W. by S., 41 miles; 6th, E.N.E., 66 miles. Required her present latitude, the distance made good, and the direct course from the place left.

The direct course is due east, and distance 70·2 miles, the ship being in the same latitude at the end as at the beginning of the traverse.

4. A ship from lat. 41° 12′ N., sails S.W by W. 21 miles; S.W. ½ S. 31 miles; W.S.W. ½ S., 16 miles; S. ¾ E., 18 miles; S.W. ¼ W., 14 miles; and W. ½ N., 30 miles; required the latitude of the place arrived at, and the direct course and distance.

Lat. 40° 5′ N.; course S. 52° 49′ W.; distance 111·7 miles.

5. A ship runs the following courses, viz.

1st, S.E., 40 miles; 2d, N.E., 28 miles; 3d, S.W. by W., 52 miles; 4th, N.W. by W., 30 miles; 5th, S.S.E., 36 miles; 6th, S.E. by E., 58 miles; required the direct course and distance made good. Direct course S. 25° 42′ E., or S.S.E. ¼ E. nearly; distance 95·69 miles.

These examples will sufficiently illustrate the principles of plane sailing, in which, course, distance, difference of latitude, and departure, are the only quantities which enter into the problem, two of them being always given. The determination of the difference of longitude made on any course, which is the distance between the meridians measured on the equator, cannot be effected by these principles, for this element is not the same as if the meridians were all parallel to each other, as is the case with the other elements. The finding of the difference of longitude is the easiest when the ship sails due east or due west, that is, upon a parallel of latitude; this is called

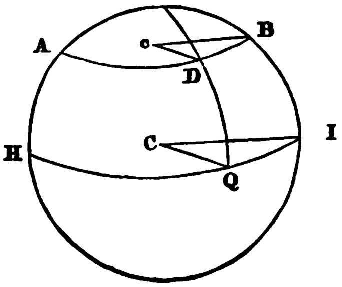
PARALLEL SAILING.

99. The theory of parallel sailing is comprehended in the following proposition, which admits of a variety of other applications.

The arc of a great circle comprehended between two of its secondaries is to the arc of a parallel small circle, comprehended between the same secondaries, as radius unity is to the cosine of the distance between the great circle and its parallel, measured on one of the secondaries. (See Spherical Geom., Prob. 2, Cor. 6.)

Applied to the case under consideration the above proposition would run as follows, viz:

The cosine of the latitude of the parallel is to the distance run as the radius to the difference of longitude. This may be demonstrated as follows:



Let IQH represent the equator, and BDA any parallel of latitude; CI will be the radius of the equator, and cB the radius of the parallel. Let BD be the distance sailed, then the difference of longitude will be measured by the arc IQ of the equator, and since similar arcs are to each as the radii of the circles to which they belong, we have

cB : CI : : dist. BD : diff. long. IQ

But cB is the cosine of the latitude IB to the radius CI, and as cosine and radius are proportional in different circles,

cB : CI : : cos lat. : R

The first two terms of these proportions being the same, the last are proportional, and we have

cos lat. : Rad. : : distance : diff. long. (1)

Corollary: hence if the distance between any two meridians, measured on a parallel in latitude L be D, and the distance of the same meridians, measured on a parallel, in latitude L' be D', we shall have (Spher. Geom., Prop. II., Cor. 6),

cos L : D : : cos L' : D' (2)

for both the ratios of (2) will be equal to R : diff. long.

By referring to proportion (1) it will be seen that if any one of the legs of a right-angled triangle represent the distance run on any parallel, and the adjacent acute angle be equal to the degrees of lat. of that parallel, then the hypothenuse will represent the difference of longitude, since this hypothenuse will be determined by that proportion.

The right angled triangle used in plane sailing may therefore be employed here, changing the names of its elements, viz., *course* into *latitude*, *difference* of *latitude* into *distance*, and *distance* into *difference of longitude*.

And a traverse table computed to degrees and fractions of a degree instead of points and quarter points, may be employed to solve problems in parallel sailing.

Formula (1) above may be expressed by the following rule. *Divide the distance sailed by the cosine of the latitude, and the quotient will be the difference of longitude.*

EXAMPLES.

1. A ship from latitude 53° 56′ N., longitude 10° 18′ E., has sailed due west, 236 miles : required her present longitude.

By the rule

lat.	53° 56	log. cos	9·76991
dist.	236	log.	2·37291
diff. long.	400	log.	2·60300

Long. left	10° 18′ E.
Diff. long. = $\frac{400}{60}$ degrees =	6 40 W.
Long. reached	3 38 E.

2. If a ship sail E. 126 miles, from the North Cape, in lat. 70° 10′ N., and then due N., till she reaches lat. 73° 26′ N.; how far must she sail W. to reach the meridian of the North Cape?

Here the ship sails on two parallels of latitude, first on the parallel of 71° 10′, and then on the parallel of 73° 26′, and makes the same difference of longitude on each parallel. Hence by the corollary,

As cos. lat.	71° 10′	arith. comp.	0·49104
: distance	126		2·10037
: : cos lat.	73 26		9·45504
: distance	111·3		2·04645

3. A ship in latitude 32° N. sails due east, till her difference of longitude is 384 miles; required the distance run.

325·6 miles.

4. If two ships in latitude 44° 30′ N., distant from each other 216 miles, should both sail directly south till their distance is 256 miles, what latitude would they arrive at?

32° 17′ N.

5. Two ships in the parallel of 47° 54′ N., have 9° 35′ difference of longitude, and they both sail directly south, a distance of 836 miles:

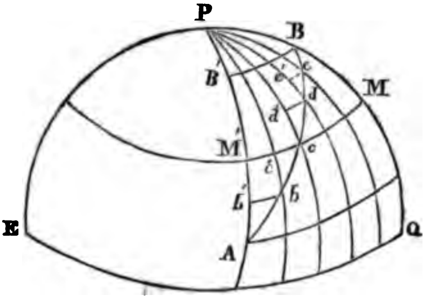
required their distance from each other at the parallel left, and at that reached.

385·5 miles, and 4[illegible]·9 miles.

MIDDLE LATITUDE SAILING.

100. Having seen how the longitude which a ship makes when sailing on a parallel of latitude may be determined, we come now to examine the more general problem, viz., to find the longitude a ship makes when sailing upon any oblique rhumb.

There are two methods of solving this problem, the one by what is called *middle latitude sailing*, and the other by *Mercator's sailing*. The first of these methods is confined in its application, and is moreover somewhat inaccurate even where applicable; the second is perfectly general, and rigorously true; but still there are cases in which it is advisable to employ the method of middle latitude sailing, in preference to that of Mercator's sailing; it is, therefore, proper that middle latitude sailing should be explained, especially since, by means of a correction to be hereafter noticed, the usual inaccuracy of this method may be rectified.



Middle latitude sailing proceeds on the supposition that the departure or sum of all the meridional distances *bb'*, *cc'*, *dd'*, &c. from A to B, is equal to the distance M'M of the meridians of A and B, measured on the middle parallel of latitude between A and B.

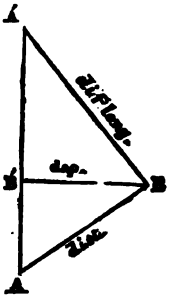
This supposition becomes very inaccurate when the course is small, and the distance run great; for it is plain that the middle latitude distance between the extreme meridians will be much greater than the departure, if the track A B cuts the successive meridians at a very small angle.

The principle approaches nearer to accuracy as the angle A of the course increases, because then as but little advance is made in latitude, the several component departures lie more in the immediate vicinity of the middle latitude parallel. But since in very high latitudes, a small advance in latitude makes a considerable difference in meridional distance, this principle is not to be recommended in such latitudes if much accuracy is required.

By means, however, of a small table of corrections, constructed by Mr. WORKMAN, the imperfections of the middle latitude method may be removed, and the result of it rendered in all cases accurate. This table we have given at the end of the present volume.

The rules for middle latitude sailing may be thus deduced.

It has been seen at (Art. 97), that the difference of latitude, departure, and distance sailed on any oblique rhumb, may be all accurately represented by the sides AB′, B′B, AB, of a right angled plane triangle. Now, by the present hypothesis, the departure B′B is equal to the middle latitude distance between the meridians of the places sailed from, and arrived at, so that the difference of longitude of the two places of the ship is the same as if it had sailed the distance B′B on the middle latitude parallel; the determination of this difference of longitude is, therefore, reduced to a case of parallel sailing; and since, as we have seen (p. 215), the formula for parallel sailing is a proportion which expresses the relation between the elements of a right angled plane triangle in which the base is the dist. sailed, the angle at the base the lat., and the hypothenuse the diff. of long., let B′BA′ be this triangle, in which, according to the theory of mid. lat. sailing, the departure B′B takes the place of the dist. sailed. From these triangles, the two partial ones of which are right angled, and the total one not, we have the following theorems, viz., in the triangle A′B′B,



cos A′BB′ : BB′ : : radius : BA′

that is,

I. Cos. mid. lat. : departure : : radius : diff. of long.

In the triangle A′BA, which is not right angled,

sin A′ : AB : : sin A : A′B ;

that is,

II. Cos mid. lat. : distance : sin course : diff. long.

In the triangle ABB′, we have the proportion (Art. 41),

R : tan A : : AB′ : : BB′

comparing this with the first proportion above, observing that the extremes of this are the means of that, we have

AB′ : A′B : : cos A′BB′ : tan A ;

that is,

III. Diff. lat. : diff. long. : : cos mid. lat. : tan course.

These three proportions comprise the theory of middle latitude sailing, and when to the middle latitude the proper correction, taken from Mr. Workman's table, is added, these theorems will be rendered strictly accurate.

This is Table XXIX; the middle latitude is to be found in the first column to the left; in a horizontal line with which, and under the given difference of latitude, is inserted the proper correction to be *added* to the middle latitude to obtain the latitude in which the meridian distance is accurately equal to the departure. The formula for constructing this table is obtained as follows·*

Let

d = proper diff. of lat.
D = meridional diff. of lat.
m = middle latitude.
$\mathrm{M} = m +$ correction.
L = diff. of longitude.

Then (Art. 100, Form III.),

$$\tan \text{course} = \frac{\cos \mathrm{M} \times \mathrm{L}}{d}$$

But (Art. 101, Rule 1),

$$\tan \text{course} = \frac{\text{rad} \times \mathrm{L}}{\mathrm{D}}$$

$$\therefore \frac{\cos \mathrm{M} \times \mathrm{L}}{d} = \frac{\text{rad} \times \mathrm{L}}{\mathrm{D}} \therefore \cos \mathrm{M} = \frac{\text{rad}\, d}{\mathrm{D}}$$

$$\therefore \text{correction} = \cos^{-1} \frac{\text{rad.}\, d}{\mathrm{D}} - m$$

EXAMPLES.

1. A ship, in latitude 51° 18′ N., longitude 22° 6′ W., has sailed S. 33° 5′ E., required her latitude and longitude.

The required latitude is found by plane sailing, as follows:

Course	33° 5′	log. cos	9·92318
Dist.	1024	log.	3·01030
Diff. Lat.	858		2·93348

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* The investigation of this formula should be postponed until after reading the next article, and may be omitted entirely.

Lat. left	51° 18′ N.
Diff. lat.	14 18
Lat. required	37° N.

To find the longitude by mid. lat. sailing.

Lat. left	51° 18′
" reached	37°
Sum	88 18
½ Sum	44° 9′ mid. lat.

Then by the proportion III. above,

cos. mid. lat. 44° 9′ ar. comp	log. 0·14418
: tan course 33 5	log. 9·81390
: : diff. lat. 858	log. 2·93349
: diff. long. 779	log. 2·89157

In this operation the middle latitude has not been corrected, so that the difference of longitude here determined is not without error. To find the proper correction, look for the given middle latitude, viz., 44° 9′ in the table of corrections, the nearest to which we find to be 45°; against this and under 14° diff. of lat. we find 27′, also under 15° we find 31′, the difference between the two being 4′; hence corresponding to 14° 18′ the correction will be about 28′. Hence the corrected middle latitude is 44° 37′, therefore,

cos. corrected mid. lat. 44° 37′ ar. comp. log.	0·14763
: tan course 33 5	9·81390
: : diff. lat. 858	2·93349
: diff. long. 785·3	2·89502

therefore, the error in the former result is about 6½ miles.

Long. left,	22° 6′ W.
Diff. of long. 785 =	13° 5′
Long. required.	9° 1′ W.

2. A ship sails in the N.W. quarter, 248 miles, till her departure is 135 miles, and her difference of longitude 310 miles; required her course, the latitude left, and the latitude come to.

Course N. 32° 59′ W.; lat. left 62° 27′ N.; lat. in 65° 52′ N.

3. A ship, from latitude 37° N., longitude 9° 2′ W., having sailed between the N. and W., 1027 miles, reckons that she has made 564 miles of departure; what was her direct course, and the latitude and longitude reached?

Course N. 33° 19′ W. or N.W. by N. nearly; lat. 51° 18′ N.; long. 22° 8′ W.

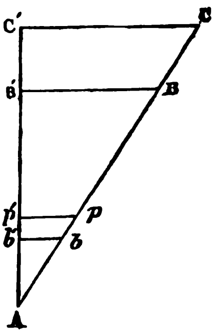
4. Required the course and distance from a point in lat. 37° 48′ N., long. 25° 13′ W., to a point in lat. 50° 13′ N., long. 3° 38′ W., the middle latitude being corrected by Workman's Table.

Course N. 51° 11′ E.; distance 1189 miles.

MERCATOR'S SAILING.

This is for the determination of difference of longitude when a ship sails on any oblique rhumb.

101. It has already been seen that when a ship sails on any oblique rhumb, the difference of latitude, the departure, and the distance run, are truly represented by the sides of a right-angled plane triangle. Let AB′B in the annexed diagram be this triangle, A representing the course, AB′ the diff. of lat., and B B the departure. Let AC′ be a sufficiently greater difference of latitude to make the corresponding departure CC′ equal to the difference of longitude required. This increased difference of latitude AC′ is called the *meridional difference of latitude*, AB′ being called the *proper* difference of latitude, by way of distinction. The solution of the triangle AC′C then will serve to determine the difference of longitude C C. In this triangle we know the course A, and we shall now show how to construct a table for finding the side AC, the meridional difference of latitude. The departure B B represents the sum of all the very small meridian distances, or elementary departures, *b′b*, *c′c*, &c., in the diagram at Art. 100, the difference of latitude AB′ represents the sum of all the corresponding small differences in the figure referred to, and the distance AB the sum of all the corresponding distances A*b*, *bc*, *cd*, &c., and each of these elements is supposed to be taken so exceedingly small as to form on the sphere a series of triangles, differing insensibly from plane triangles.



Let A*b′b* in the annexed diagram represent one of these elementary triangles, *b′b* will be one of the elements of the departure, and A*b′*, the corresponding difference of latitude; and as $b'b$ is a small portion of a parallel of latitude, it will be to a similar portion of the equator, or of the meridian, as the cosine of its latitude to radius (Art. 99), this similar portion of the equator, or of the meridian, being the difference of longitude between b' and b. Suppose now the distance Ab prolonged to p, till the departure $p'p$ is equal to the difference of longitude of b', and b, then $b'b$ will be to $p'p$ as the cosine of the latitude of $b'b$ to the radius; but $b'b : p'p :: \text{A}b' : \text{A}p'$; hence the proper difference of Ab' is to the increased difference Ap' as the cosine of the latitude of $b'b$ to the radius. Calling, therefore, the proper difference of latitude d, the increased difference of latitude D, the latitude of $b'b$, l, and the radius 1, which it is in the table of natural sines, this proportion will be in symbols,

$$d : \text{D} :: \cos l : 1$$

$$\therefore \text{D} = \frac{d}{\cos l} = d \sec l \text{ since } \sec = \frac{1}{\cos} \quad \text{(Art. 33.)}$$

The ship, therefore, having made the small departure $b'b$, and the difference of latitude Ab, must continue her course till the difference of latitude becomes D, in order that her departure may become equal to the difference of longitude corresponding to $b'b$. Conceiving all the elementary distances to be in this manner increased, the sum of all the corresponding increased departures will necessarily be the whole difference of longitude made by the ship during the course. The determination of AC′ requires the previous determination of all its elementary parts; if d be taken equal to 1′, each of these parts will be expressed by $\text{D} = 1' \sec l$, or $\text{D} = \sec l$, that is, $\sec l$ expresses the meridional difference of latitude corresponding to a proper difference of latitude of 1′ at the latitude l, giving l successively the values 1′, 2′, 3′, &c., up to 90°, and adding the result of the second substitution to that of the first, and so on, we shall have in succession, the values of the increased latitude corresponding to 1′, 2′, 3′, &c. of proper latitude; these values are called the *meridional parts*, corresponding to the several proper latitudes, and when registered in a table, form a table of meridional parts, given in all books on Navigation.*

The following scheme may serve as a specimen of the manner in which such a table may be constructed, and, indeed, of the manner in which the first table of meridional parts was actually formed by Mr. Wright, the proposer of this ingenious and valuable method.

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* In other words, a table of meridional parts is a table of differences of latitude expressed in geographic miles, each difference of latitude being enough greater than its corresponding proper difference of latitude, to make the departure equal to the difference of longitude. The table gives the meridional difference between the equator and any given latitude.

Mer. parts of 1′= nat. sec 1′.
Mer. parts of 2′= nat. sec 1′+ nat. sec 2′.*
Mer. parts of 3′= nat. sec 1′+ nat. sec 2′+ nat. sec 3′.
Mer. parts of 4′= nat. sec 1′+ nat. sec 2′+ nat. sec 3′+ nat. sec 4′, &c., &c.

Hence, by means of a table of natural secants, we have

			Nat. secs.		Mer. parts.
Mer. parts of 1′=			1·0000000		1·10000000
Mer. parts of 2′=	1·0000000	+	1·0000002	=	2·00000002
Mer. parts of 3′=	2·0000002	+	1·0000004	=	3·00000006
Mer. parts of 4′=	3·0000006	+	1·0000007	=	4·00000013
&c.					&c.

There are other methods of construction, but this is the most simple and obvious. The meridional parts thus determined are all expressed in geographical miles, because in the general expression D = 1′ sec l, 1′ is a geographical mile.

Having thus formed a table of meridional parts (Table III. at the end), if we enter it with the latitudes sailed from, and reached, and take the difference of the corresponding parts in the table, the remainder will be the meridional difference of latitude, or the line AC′ in the preceding diagram.

The angle A is the given course, so that there are known in a right angled triangle AC′C, an angle and the side adjacent, to find the side opposite. The following is the rule.

1. *The tangent of the course* × *meridional difference of latitude = the difference of longitude;* or if the departure be given instead of the course, then from the similar triangles AB′B, AC′C, the proportion will be

2. *As the proper difference of latitude is to the departure, so is the meridional difference of latitude to the difference of longitude.* Other proportions immediately suggest themselves from the preceding figure.

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* Observe that the meridional parts, or meridional diff. of latitude from the equator to 2′ of latitude will be the sum of the meridional parts from the equator to 1′, plus the meridional parts from 1′ to 2′, which latter is nat. sec. 2′. Again, that the meridional parts from the equator to 3′ is the sum of the meridional parts from the equator to 2′, and the meridional parts from 2′ to 3′, which latter is the nat. sec. 3′.

EXAMPLES.

1. A ship from latitude 42° 31′ N., and longitude 58° 51 W., having sailed S.W. by S. 300 miles, required the latitude and longitude at which she has arrived.

To find the diff. of lat. by plane sailing.

Dist.	300	log.	2·47712
Course,	33° 45′	log. cos	9·91985
Proper dif. lat.	249·4	log.	2·39697

To find the diff. of long. by Mercator's sailing.

Lat. left,	42° 30′ N*,	mer. parts	2822
Prop. diff. lat. 249 =	4° 9′		
Lat. reached,	38° 21′*	mer. parts	2495

Long. left,	58° 51′	Mer. diff. lat.	327	log.	2·51455
Diff. long. 219 =	3° 39′	Course,	33° 45′	log. tan	9·82489
Long. reached,	62° 30′	Diff. of long.	218·5	log.	2·33944

2. Required the course and distance from Cape Cod light-house, lat. 42° 3′ N., long 70° 4′ W., to the island of St. Mary's, lat. 36° 59′ N., long. 25° 10′ W.

Cape Cod, lat.	42° 3′ N.	Mer. parts,	2786	Long.	70° 4′ W.
St. Mary's, lat.	36° 59′ N.	Mer. parts,	2391	Long.	25° 10′ W.
	5° 4′	Mer diff. lat.	395		44° 54′
	60				60
Prop. diff. lat.	304			Diff. long.	2694 miles.

To find the Course (by Rule 1.)		To find distance by Plane Sailing.	
Merid. diff. lat. 395 log.	2·59660		
Diff. of long. 2694 log.	3·43040		
Course 81° 40′ log. tan	10·83380	log. cos.	9·16116
Proper diff. lat. 304 miles		log.	2·48287
Distance 2098 miles		log.	3·32171

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* Enter tab. of mer. parts with the argument 42° 30′, and take out the mer. parts corresponding, viz., 2822, which is the meridional diff. of lat. from the equator to 42° 30′ N. Again enter the tab. with the argument 38° 21′; the corresponding mer. parts, or mer. diff. of lat., from the equator to lat. 38° 21′ N. will be found to be 2495. Subtract the latter from the former, and the remainder will be the meridional diff. of lat. between 38° 21′ N., and 42° 30′ N.

3. A ship from latitude 37° N., and longitude 32° 16′ W., has sailed in a north-westerly direction 300 miles, till she has reached the latitude of 41° N. Required the precise course on which she has sailed, and the longitude at which she has arrived.

Ans. Course 36° 52′; Long. 36° 8′

APPENDIX III.

GREAT CIRCLE SAILING.

1. The shortest path from one point to another on the surface of a sphere is the arc of a great circle (Geom., App. III., p. 2). A ship, therefore, sailing on the arc of a great circle, joining her point of departure and point of destination on the surface of the earth, will make a shorter voyage than if she sails in the direct course, that is upon the rhumb line joining the same two points.

The practical application of great circle sailing will consist in determining as often as the ship's place is found, that is to say her latitude and longitude, which, under ordinary circumstances, occurs daily, the direction which she ought to take, in order to sail on the great circle from the point where the ship is, to the point of destination. This problem is, in effect, solved in the example on p. 137. In the diagram at that place, S denotes the point where the ship is, S′ the point of destination, and the angle PSS′ the course which the ship ought to steer, in order to sail on the great circle from S to S′. The solution of the problem of Great Circle Sailing, it appears, from the example referred to, requires the application of Napier's Analogies, forms IX. and X. of Art. 86. In a practical treatise on Great Circle Sailing, which appeared in 1846, by S. T. Coit, a table will be found called the "Great Circle Table." It is a table of double entry, in which the logarithm of the ratio of the cosine of the half sum, to the cosine of the half difference, and of the ratio of the sine of the half sum to the sine of the half difference of the colatitudes of S and S′, will be found computed for any two latitudes. You enter this table with the less of the given latitudes at top, and the greater at the side; under the former, and on the range of the latter, in the column entitled sine, is found the logarithm of the ratio of the sine of half the sum to the sine of the half difference, and in the column entitled cosine, the ratio of the cosine of the half sum to the cosine of the half difference; if to each of these be added the log. cotangent of the difference of longitude of the point where the ship is, and the point for which she is destined, the results will be the logarithmic tangent of the half sum and half difference of the angles S and S′, the former of which, viz., S will be the course upon which the ship should be directed.

The example at p. 137 adapted to this place should read as follows:

1. A ship from lat. 20°, long. 41° 34′ 26″, is bound to a point in lat. 51° 30′, long. 10°, upon what course must she sail in order to pursue the shortest path to her destination?

Ans. 30° 28′ 12″.

2. A ship in lat. 40° 30′ N., long. 70° W., is bound for a place in lat. 51° 22′ N., long. 9° 37′ W., required the course for Great Circle Sailing?

Ans. N. 54° 3′ E.

The computation of the third side ss′ in the same triangle gives the distance sailed.

As a steamer in ordinary weather pursues steadily the course of the great circle from port to port, it may be convenient to calculate beforehand the position of the points in which this great circle intersects the meridians for every five degrees of longitude (five degrees being about the daily progress of a first class steamer), and then if the ship lays upon the direct course for these points successively, it will be sufficient, since the rhumb line differs insensibly from the arc of a great circle for so short a distance. The method of determining these is simple. For when the angle s is calculated, it is only necessary to employ the last two forms, XI. and XII. (Art. 86) of Napier's Analogies, which are used for solving a spherical triangle when two angles and the interjacent side are given.

The data will be the colatitude of the point s, the course Pss′ previously calculated, and the angle which the meridian Ps makes with the meridian whose point of intersection with the great circle course from s to s′ is to be calculated; in other words, the difference of longitude of these two meridians.

The logarithm of the ratio of the cosines and sines of the half sum and half difference of the given angles may be taken from "The Circle Table," entering it with the complements of these angles; the log. tangent of the given side will be the same in the calculation for each meridian; the solution of the triangle for each meridian giving the colatitude of the point in which the great circle path intersects the meridian, and the course which the ship ought to take in departing from that meridian.

EXAMPLE.

A ship sailing from the port of New York to Havre or Liverpool, by the shortest path, would steer from Sandy Hook, lat. 40° 27′ 30″, long. 74° 00′ 48″ W., E. ½ S.[*] running on the rhumb line, and thus give the south shoal of Nantucket a berth of about 15 miles; from this she would sail to a point in lat. 41°, long. 68°, on southern part of George's shoal, in 25 fathoms of water. From this point she would commence Great Circle Sailing, nothing being gained in taking the great circle rather than the rhumb line in the previous short distances. As the great circle from this point to that of destination would pass over Newfoundland, it is necessary to divide the voyage between two great circles, the first terminating at Cape Race, and the second terminating at Cape Clear, the south point of Ireland.

Required the course from the south shoal of Nantucket to George's Shoal, and the points at which the great circle from lat. 41° N., long. 68°, W., to Cape Race, in lat. 46°, 39′ 24″, long. 53° 04′ 36″, intersects the meridians of 60° and 55° W.; and the points in which the great circle from Cape Race to Cape Clear, in lat. 51° 22′ N., long. 9° 37′ W., intersects the meridians of 45°, 40°, 35°, 30°, 25°, 20°, and 15° W.

Ans. The course from Nantucket to George's is

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[*] This allows for variation of compass.

The first great circle intersects the meridian of 60° at 44° 24′ 25″, of 55° at 46° 5′ 28″ of N. latitude.

The second great circle intersects the meridian of 45° at 48° 55′ 47·″5, of 40° at 49° 58′ 27″, of 35° at 50° 46′ 6″, of 30° at 51° 19′ 44″, of 25° at 51° 40′ 3″, of 20° at 51° 47′ 28″, and of 15° at 51° 42′ 7″.*

The following, from Mr. Coit's work, to which the student is referred for a variety of useful problems, and interesting information, will furnish a number of exercises.

TRACK OF THE ARC OF A GREAT CIRCLE FROM CHESAPEAKE BAY TO BORDEAUX.

Table showing the longitude of the intersections of the latitudes crossed by the arc of a great circle from Cape Henry's light-house, mouth of Chesapeake Bay, to the Corduan light, near Bordeaux; also showing the courses from the points of intersections of the latitudes crossed, and from intersection to intersection.

	Latitudes crossed	Long. of the intersec. of the lat. crossed.		Courses towards Corduan from the points of intersection.	Courses towards Cape Henry from the points of intersection.
		From the Meridian of Greenwich.	From the Meridian of Corduan.		
C. Henry.	36°·56′	76°·04′	74°·54′	N. 55°·40′ E.	. . . .
	37 ·	75 ·55	74 ·45	– 55 ·46 –	S. 55°·46′ W.
	38 ·	74 ·02	72 ·52	– 56 ·55 –	– 56 ·55 –
	39 ·	72 ·01	70 ·51	– 58 ·10 –	– 58 ·10 –
	40 ·	69 ·52	68 ·42	– 59 ·32 –	– 59 ·32 –
	41 ·	67 ·36	66 ·26	– 61 ·01 –	– 61 ·01 –
	42 ·	64 ·25	63 ·15	– 62 ·41 –	– 62 ·41 –
	43 ·	62 ·22	61 ·12	– 94 ·31 –	– 64 ·61 –
	44 ·	59 ·19	58 ·09	– 66 ·37 –	– 66 ·37 –
	45 ·	56 ·27	55 ·17	– 69 ·03 –	– 69 ·01 –
	46 ·	51 ·52	50 ·42	– 71 ·53 –	– 71··53 –
	47 ·	46 ·55	45 ·45	– 75 ·29 –	– 75 ·29 –
	48 ·	39 ·56	38 ·46	– 80 ·38 –	– 80 ·38 –
Max. Lat.	48 ·41	27 ·25	26 ·15	East.	West.
	48 ·	14 ·54	13 ·44	S. 80 ·38 E.	N. 80 ·38 W.
	47 ·	7 ·55	6 ·45	– 75 ·29 –	– 75 .29 –
	46 ·	2 ·58	1 ·48	– 71 ·53 –	– 71 ·53 –
Corduan.	.45 ·35	1 ·10	0 ·	. . . .	– 70 ·36 –

On this track the difference or saving is 110.

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* The great circle courses and distances are as follows: From George's Shoal to the meridian of 60°, 57 14′ and 497 miles. From 60° to 55° N., 62° 40′ E., and 234 miles. From 55° to Cape Race N. 66° 13′ E., and 87 miles. From Cape Race to the meridian of 45° N. 61° 19′ E., and 353 miles. From 45° to 40°, N. 70° 18′ E., and 205 miles. From 40° to 35°, N. 74° 6′ E., and 197 miles. From 35° to 30°, N. 77° 57′ E., and 191 miles. From 30° to 25°, N. 80° 50′ E., and 188 miles. From 25° to 20°, N. 85° 45′ E., and 186 miles. From 20° to 15° N. 89° 42′ E., and 186 miles. From 15° to Cape Clear S. 86° 23′, and 202 miles.

For further information on Great Circle Sailing, see "The Practice of Navigation and Nautical Astronomy, by Lieut. Raper, of the Royal Navy," Art. 336, in which work will also be found a convenient table (tab. 5) called the Spherical Traverse table, for solving problems in this kind of sailing.

See also a small collection of "Tables to facilitate Great Circle Sailing," by John Thomas Towson, published by order of the Lords Commissioners of the Admiralty.

SUMNER'S METHOD.

This is a method discovered recently by accident, and consists in calculating the ship's longitude by chronometer for two assumed latitudes, the one of which is the next even degree less, the other the next even degree greater (without odd minutes) than the latitude by dead reckoning. The two positions of the ship thus determined from the longitudes found and the assumed latitudes, being projected on a Mercator's chart, the line joining them passes through the true position of the ship, and any land it may happen to pass through in the vicinity, will have the same bearing from the ship that this line makes with the meridian.

The theory is, that this line is a small portion of what the author terms a parallel of equal altitude, that is, a small circle of the terrestrial sphere, the pole of which is the point of the earth's surface, at which the sun is vertical or in the zenith at the instant of observation. To all places situated on this circle the sun will have the same altitude at the instant. Now since in the two computations in the above problem the latitude only is different, the altitude and declination, which are the other data, remaining the same, the altitude of the sun is therefore the same at the two positions determined, and they are in the same parallel of equal altitude, and as the observed altitude of the ship is also the same, the ship, too, is upon the same parallel of equal altitude, a small arc of which may be regarded as a straight line.

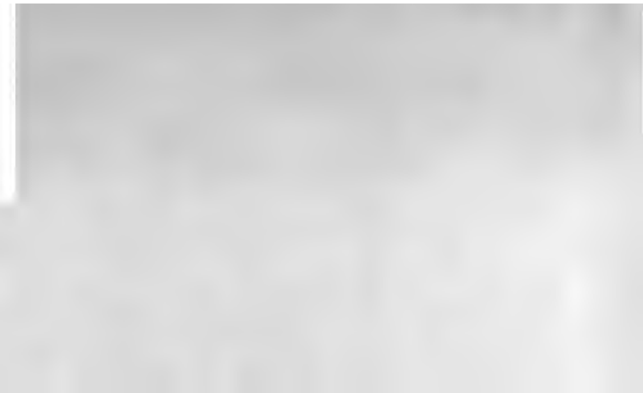
A perpendicular to the line determined as above will be in the direction of the sun's bearing, and the angle it makes with the meridian will be the sun's azimuth. For the perpendicular to an arc will pass through the pole of the arc.

If two altitudes of the sun be taken, and two lines projected as above, passing each through the place of the ship, its actual position is determined by their intersection.

For the method of allowing for the change of place of the ship between two observations for altitude, and for a variety of problems based upon the principle above explained, see Sumner's work.

PART IV.

SURVEYING.



PART IV.

SURVEYING

102. HAS for its object to make upon paper an exact delineation called a map, plot, or draft, and sometimes to find the contents of ground.

The most common mode of proceeding is to measure a base line upon the ground, as at Art. 10, and take the angles at each extremity with some instrument suitable for the purpose, thus determining the position of a distinct point. This is transferred to paper, by means of a scale of equal parts, and a semicircular protractor, as described in the same article. As many points upon the ground as may be desirable can thus be transferred to the map. If two points thus determined be the extremities of some straight boundary, as a wall, fence, or side of a building, the boundary itself is drawn by uniting the two points by a straight line. If the boundary be curved or irregular, as the bank of a stream, a coast, the border of a wood, &c., the prominent points should be determined on the map, as above described, and the boundary then traced through them with the hand, by the eye.

The positions of points on a map may also be determined by taking their direction and distance from some point used as a basis or base point for the whole survey. Or, when more convenient, some one of the points determined from the first base point may itself become a base point for the determination of others more in advance. The direction of one point from another is expressed by the angle which the line of direction makes with some known line. A very convenient line for this purpose is a meridian or north and south line, for the direction of this latter may always be known by means of a magnetic needle, and the direction of any line from a north and south line by a compass. Allowance must of course be made for what is called the variation of the needle, which is determined by astronomical observations, in a manner to be hereafter explained. Distances are measured upon the ground with a tape or chain. The measuring tape is covered with wax, to prevent the effects of varying degrees of moisture, in contracting and expanding it. It is divided usually into feet and inches. The chain is of iron wire, each link being the hundredth part of the whole chain, which is 4 rods, or 66 feet, or 792 inches, so that each link is 7·92 inches in length. Every ten links is, for convenience of counting, marked by a piece of brass, with as many tongues as the brass piece is tens of links from the extremity of the chain. A line is measured on the ground, as follows: Two persons take hold, one of each extremity of the chain, and one going in front, towards the point whose distance is to be measured, carries in his hand a staff and ten marking pins, of iron wire, each about two feet in length, sharpened to stick in the ground at the extremity of the chain measured off.

The one behind, by a motion of the hand to the right or left, indicates to the other whether the staff which is held at the end of the stretched chain is on the alignment of the distant point or not. As soon as he discovers it to be so, he makes a motion with his hand downward, and the other places a marking pin. Both then move on, the one behind taking up the marking pins which the other has left in the ground. When all the pins have been passed from one to the other, ten chains have been measured off.

For fixing the position of points of ground upon a map, the best instrument, when the survey is of moderate extent, is one which surveys and plots at the same time, called the

PLANE TABLE.

This consists of a rectangular board, mounted upon a three-legged stand, called a tripod, to which it is attached by a ball and socket movement, that is to say, there is a socket fastened to the tripod and a ball clasped by the socket, which moves in it, the ball being fastened to the underside of the table. This permits the table to be placed exactly horizontal. To ascertain whether it is so or not, a detached spirit level is placed upon the table temporarily, and the table is levelled by means of three screws, which pass through a horizontal circle of wood or brass which projects round the top of the tripod, the screws working against the table underneath. These screws are placed near the outer edge of the circle, and at distances of 120° from each other. If one of them be screwed in one direction, it lifts the table on that side; if in the opposite direction, it lets it down. In order to level the table by means of these screws the spirit level is placed over the line joining two of them, and by moving them the bubble is brought to the centre; this renders one line of the table horizontal. The spirit level is then placed in a direction perpendicular to its former one, and the bubble brought to the centre by turning the third screw, leaving the others untouched; two lines of the table are then horizontal, and consequently the table itself.* The spirit level, which is a tube of glass inclosed in one of brass, and containing spirits of wine, rests on short feet at the ends, one of which is made movable by a screw, and should be adjusted by reversing the level, end for end, on the table, after the bubble is first brought to the centre, when, if it departs from the centre, it must be brought back half by the foot screw of the level itself, and half by the levelling screws of the table. This process is to be repeated till in both positions of the level the bubble remains in the centre. A necessary appendage to the plane table is a brass ruler, with a thin edge, upon which are mounted either plain or telescopic sights, the line of vision being parallel to the edge of the ruler. Plain sights consist of two upright flat pieces of brass, one at each end of the ruler, facing each other with a narrow vertical aperture in each, to look through. Sometimes the aperture is made wider in the one towards the object, and a vertical thread or hair stretched down the middle of it, which serves for a sight. Larger orifices are made at some parts, in which to catch sight of the object, which is then brought down to a fine sight in the narrow part of the aperture.

When the sights are telescopic, the telescope may be mounted like that of a transit instrument (if the ruler is very wide) upon the upright piers of brass, by means of a small horizontal axis. Or upon a narrow ruler the telescope is supported at the top of a single column of brass, by a stout axis projecting from one side. At the focus of the object glass of the telescope two lines of spiders web cross each other exactly in the optical axis of the telescope. When the ruler is placed upon the table, by turning the table round a vertical axis called the axis of the instrument, the line of sights may be turned in any horizontal direction at pleasure. The telescope has a small vertical play also upon the horizontal axis, for the purpose of directing it to objects somewhat elevated or depressed. To use the instrument, place it over one end of the base line on the ground, so that a line to represent the base line drawn upon paper, stretched tightly and immovably upon the table, may be in the same direction. This is done by placing the edge of the ruler upon the line on the paper, and then turning the table upon the vertical axis of the instrument until a staff placed at the other end of the base line is seen upon the line of sights. There is a convenient arrangement for accomplishing this, consisting of what are termed a clamp and tangent screw, or screw of slow motion, an arrangement which is applied to almost all instruments. This consists of a screw working near its head, in a collar in which it has no longitudinal motion, the thread of the screw working in a piece of brass which is at pleasure loose from or screwed against the stand below, by a second screw, called the clamp screw, the first being called the tangent screw. When the instrument is "clamped," the tangent screw being turned, pushes the side of the table to which its collar is attached slowly away from the part of the tripod below, thus giving a slight motion to the table on its vertical axis. The line of sights being thus arranged in the direction of the base line on the ground, whilst the edge of the ruler coincides with the base line upon the paper, keeping now the edge of the ruler passing through one extremity of the latter, whilst the table is held immovable by the clamp screw, upon the tripod, turn the line of sights accurately towards one of the points to be plotted, and draw a line along the edge of the ruler upon the paper, marking it 1. Turn the line of sights successively to all the points in view to be plotted, drawing lines on the paper from the extremity of the base line on the paper, in the directions of these points, as before, and numbering them 2, 3, and so on, in order. Let the instrument now be taken up and carried to the other extremity of the base line, levelled, and the edge of the ruler being placed upon the base line on the paper, in a reversed position, bring the line of sights in the direction of the station at the first extremity of the base, by turning the table on its vertical axis, using the clamp and tangent screw, as before. Proceed in the manner just described, to draw lines also from this extremity of the base on paper, in the directions of the same objects, by sighting towards them in the same order, and numbering the lines as before. Where 1 meets 1 will be the position of the first object on the map; where 2 meets 2 the position of the second object, and so on.

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* This operation has generally to be repeated, as the levelling of the second line deranges a little the level of the first.

The telescope of the plane table has sometimes an arrangement by which distances can be measured without the use of the tape or chain. Two sets of cross wires, as the spider lines are technically called, are placed at the focus of the object glass, the points of intersection of each pair being a small distance apart, the one above the other. If a staff be placed 100 feet from the telescope, and a space of exactly one foot be seen intercepted between the intersection of the wires; if then the staff be removed to a distance of 200 feet, two feet will be seen intercepted on the staff, because, according to an optical law, the size of the image formed at the focus of the object glass is inversely as the distance of the object from the instrument, so that the space between the wires at the focus, where the image is formed, being occupied by 1 foot at the distance of

100 feet, the image of this 1 foot at the distance of 200 feet will occupy only half the space, or it will take an image of two feet to occupy the whole space.

To ascertain the proportions of the instrument by experiment, place a staff at such a distance that 1 foot may be intercepted between the intersections of the wires; measure this distance, and it will be the distance to be multiplied by the number of feet and fractions of a foot seen intercepted when the instrument is used for measuring distances. The staff used should be about two inches broad by one in thickness, and painted white, the divisions and numbers upon it being black or deep red, and made very distinct.

THE SURVEYOR'S COMPASS.

This instrument is a circular box of brass about six inches in diameter, and half an inch deep, mounted upon a tripod with ball and socket motion. The bottom of the box on the interior is silvered, and the circumference of this silvered bottom is graduated. In the centre of the bottom stands up a pivot upon which a long magnetic needle is accurately balanced. The top of the box is of glass, in order that the whole interior may be seen. Upon the box above, in the direction of a diameter, is a line of sights which may be plain or telescopic. The graduation is numbered from each end of the diameter, and runs to 90° each way.*

To survey a polygonal field with this instrument place it at one of the corners of the field, and direct the line of sights to the next corner along one of the straight boundaries of the field, and measure with a chain the length of this boundary line. Enter in a field book, ruled in three columns, in the first column the number of the station beginning with station 1; in the second column the Bearing (which would be, for instance, N. 30° E., if the line of sight were directed to the right of the north end of the needle marked with a cross, and the needle pointed to 30° on the graduated circle;†) and in a third column the distance measured with the tape or chain. Take the instrument now to the corner whose position has just been determined, calling it station number 2, and determine the position of station number 3, a third corner of the field and the bounding line connecting stations 2 and 3 in the same manner, and so proceed quite ound the field. (See p. 233.) To plot this assume the point on the paper at which you will have the first corner of the field, draw through it a line to represent a magnetic meridian, or north and south line, and another line making an angle with this, equal to the first bearing, as 30° above, which will be laid off with the semi-circular protractor to the right of the meridian line if the bearing be east, and vice versâ; then from a scale of equal parts lay down the distance taken from the third column of the field book, and this will determine the second station of the map; through this draw a north and south line parallel to the first drawn, and lay off the second boundary by its bearing and distance, in the same manner, and so proceed till the plot is complete. The accuracy of the work will be tested by the last bearing and distance, reaching exactly to the first station from which the work commenced.

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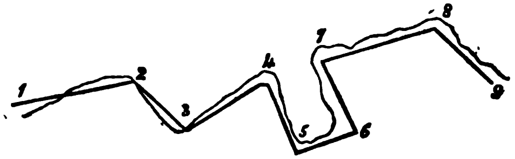
* Sometimes the graduation is numbered from 0° to 360°, the 0 and 180 diameter being the line of sights.

† If the graduation of the compass box be from 0 to 360, it will only be necessary to record the number to which the north end of the needle points.

The same method may be pursued with the plane table, in an obvious manner. It is only necessary, as the table is moved from corner to corner of the field, to place the line on paper which has just been determined parallel to the corresponding line on the ground, by placing the ruler upon it, and sighting back from the 2d to the 1st station, turning the table on its vertical axis, for the purpose. The line joining the 2d and 3d station may then be drawn by sighting to the 3d station, the edge of the ruler passing through the 2d station in the paper, and by a scale of equal parts the length of this line laid down, and so on.

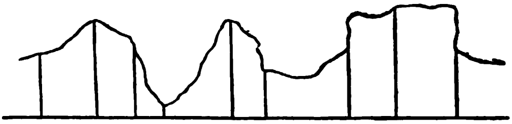
The compass may be employed to survey an irregular line, as a road, the border of a stream, a wood, a coast, &c., by taking stations sufficiently numerous to include portions nearly straight between them.

Thus:



Another mode is to run a straight line along the irregular boundary, and measure *offsets*, that is, perpendiculars to the main line, extending to the boundary at points where there are remarkable changes.

Thus:



The perpendiculars not only are measured with the chain, but the distances between them. To plot such a piece of surveying, the main line having been laid down on paper by its bearing observed with the compass, the measured distances between the perpendiculars are laid off on this line by a scale of equal parts, the perpendiculars drawn and laid off by the same scale, and the boundary traced by the hand through their extremities. An instrument for taking offsets is the surveyor's cross, consisting of two pairs of plain sights, at right angles to each other. Place the cross on the principal line, sight with one pair to a distant staff upon it, the other pair will be directed in a perpendicular to the principal line. Where a survey is extensive, the relative positions of distant points are fixed with an instrument of greater accuracy for taking angles than any we have yet described, by a process similar to that mentioned at p. 236, called *triangulation*. The instrument used for this purpose, of which that described at Art. 10 is a rude imitation, is called

THE THEODOLITE.

This consists of a horizontal circle of brass resting upon a tripod by three foot-screws or levelling screws. The circumference of the circle is silvered, and divided into degrees and parts of a degree, numbered from 0 to 360. Directly above a diameter of this circle is supported a small telescope, to which is attached a vertical circle, the plane of which is parallel to its optical axis. There are various modes of supporting the telescope. The best is by means of two upright columns of brass, standing upon a horizontal circle concentric with the horizontal limb, and moving upon or within it, round a vertical axis.* Between these columns the telescope is suspended, by means of two projections from the sides of its tubes, resembling the trunnions of a cannon, the ends of which rest in notches in the tops of the upright columns, called Y'. These projections are called the supporting axis, and this passes through the centre of the vertical circle, which is firmly attached to it at one end, and which revolves with this axis. The circumference of the vertical circle is also graduated, and called the vertical limb, the numbers on this limb running from 0 to 90° four times.

The indices which show the number of degrees passed on the horizontal limb in a horizontal direction, or on the vertical limb in a vertical direction, thus indicating the angular motion of the line of vision, or optical axis of the telescope, are of peculiar construction, and called *verniers*, from the name of their inventor. As they are used upon many instruments, we shall presently describe them in detail.

* This mounting is similar to that of the transit instrument.

There are usually three of these upon the horizontal limb, and two upon the vertical, at distances from each other of 120° in the one, and 180° on the other. The object of having more than one is to correct for wrong centring of the circle, or excentricity. Those which apply to the horizontal limb are attached to the concentric movable circle which supports the telescope, and move with it. A clamp screw serves to fasten this circle to the limb, and a tangent screw to give it a slow motion along the limb. Those which belong to the vertical limb are attached to the supports of the Y's, immovably except for adjustment, the limb moving past them with the telescope about the supporting axis.

The line of sights is marked by two wires, one vertical, the other horizontal, crossing each other at the focus of the object glass of the telescope.

These wires are fastened across a diaphragm smaller in diameter than the tube of the telescope, and held within it by means of screws passing through from the outside of the tube on the right and left. By tightening the right, and loosening the left, or vice versâ, the diaphragm may be moved laterally. The instrument is used for measuring either horizontal or vertical angles. A horizontal angle is one formed by two lines in a horizontal plane; or it is the angle included between two vertical planes which meet. A vertical angle is one formed by two lines in a vertical plane. An angle of elevation is a vertical angle formed by a horizontal line, with an oblique line coming from above to meet it. An angle of depression is a vertical angle formed by a horizontal, with an oblique line meeting it from below. For the measurement of such angles it is evident that the axis about which the verniers of the horizontal limb move, which is called the axis of the instrument, should be truly vertical, that the supporting axis on which the vertical limb turns, should be truly horizontal, and that the line of vision of the telescope should be exactly perpendicular to the latter. The processes of placing them so are called adjustments. The last mentioned, which is the first in order to be made, is called *collimation.* This is accomplished by placing the intersection of the wires upon some distant, well-defined point, then reversing the supporting axis in the Y's, by turning it end for end. If the intersection of the wires passes through the same object, as the telescope is turned round the supporting axis, the instrument is collimated, and no adjustment is necessary. But if not, the intersection of the wires must be brought half way, by estimation, towards the object by means of the screws of the diaphragm, and the other half by the tangent screw of the horizontal limb. This process must be repeated, owing to the difficulty of estimating just half, till in both positions of the supporting axis the intersection of the wires passes through the same distant point.* The second adjustment in order, consists in rendering the supporting axis horizontal. This is accomplished by means of a spirit level, composed of a glass tube, not quite filled with alcohol, leaving an air bubble at top. This tube is partly encased in a brass one, which rests by its ends upon two feet, notched at the bottom, to stand upon the ends of the supporting axis, so that the tube of the level is above, and parallel to the supporting axis, striding over the telescope, and hence the name of striding level, by which it is known. First, to adjust the axis of the level tube into parallelism with the supporting axis, when the feet of the striding level rest upon its extremities. Bring the bubble to the centre by the levelling screws of the instrument; reverse the level upon the axis, turning it end for end; if the bubble does not continue in the centre, make half the correction by the levelling screws, the other half by filing away the notch in one of the feet, viz., that nearest to the bubble, as being the highest. Sometimes the foot of the striding level is made capable of being lengthened or shortened, by means of a screw, by turning which the adjustment may be made. The parallelism between the axis of the spirit level and the supporting axis being established, whenever the bubble is at the centre, the latter of these axes is horizontal. To make the supporting axis perpendicular to the vertical axis of the instrument, bring the bubble of the level to the centre, by the levelling screws, turn the instrument round the vertical axis 180°; if the bubble now departs from the centre, make half the correction by the levelling screws, the other half by screws which elevate or depress one of the Y'.

The above are the principal adjustments. To measure an angle, with the instrument thus adjusted, one plane passing through the vertical axis of the instrument, and perpendicular to the supporting axes, is made vertical by bringing the bubble to the centre in one position; then turning the instrument round the vertical axis 90°, bring the bubble again to the centre, and the vertical axis will be truly vertical, because it is perpendicular to two horizontal lines which intersect, viz., the lines determined by the spirit level, and consequently to a horizontal plane. The instrument is now prepared for the measure of either a horizontal or vertical angle, having its vertex at the centre of motion of the instrument. For the former, turn the telescope in the direction of one of the sides of the angle, and take the reading of the degrees from one of the three verniers on the horizontal limb (which are engraved each with one of the letters A, B, C), and of the minutes and seconds from all three. A mean of the minutes and seconds is taken by adding up the three readings for the minutes and seconds, and dividing the sum by three. The telescope is now turned in the direction of the other side of the angle, by sighting to a station at its extremity, and taking the reading from the three verniers as before. The difference between this reading and the former one will give either the angle required in degrees, minutes, and seconds, or else a number which, subtracted from 360°, will give the required angle. As for instance, if the vernier stood first at 350°, moving a short distance past 360° to 10°, the angle passed over would be only 20°, which is obtained by subtracting the difference between 350° and 10°, or 340° from 360°.

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* For the theory of this process see transit instrument, p. 149, note.

In some theodolites the horizontal limb itself as well as the vernier circle, has a motion round the vertical axis of the instrument, and a clamp and tangent screw. This is for the purpose of introducing the repeating process into the measurement of horizontal angles, which is conducted as follows. Sight first to the object in the direction of one of the sides of the angle to be measured, and take the reading; loosen the clamp screw of the vernier circle, and turn this with the telescope round the vertical axis of the instrument to sight to the second object in the direction of the other side of the angle, using the clamp and tangent screw of the vernier circle for making the sight exact. Loosen now the clamp screw of the limb, keeping that of the vernier circle tight, and bring the telescope back to the first object, clamping the limb there, and adjusting the sight by the tangent screw of the limb. Repeat this process several times, ending with a movement of the vernier circle to bring the telescope upon the second object; take now the reading; the difference of the two readings divided by the number of times that the vernier circle has been moved forward, say six times, will be the angle required, with a probable error of observation of $\frac{1}{6}$ what it would have been without the repeating process.

To measure a vertical angle with the theodolite, level the instrument, or make its vertical axis truly vertical as before; elevate or depress the telescope to the object, in the direction of the inclined side of the vertical angle, and take the reading by both verniers of the vertical limb. Turn the instrument on its vertical axis 180°, and bring the telescope to bear upon the same point as before, taking the mean of the two verniers again. Half the sum of the mean of the results of the readings in the two positions of the telescope will be the angle of elevation, or depression, as the case may be corrected for index error.* Half the difference of the same

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* Because if the index be a little *behind* the zero on the limb when the telescope is horizontal in the one position, it will be a little *before* the zero in the reverse position, after turning the instrument 180° on its vertical axis.

same results will be the index error which must be applied as a correction if the reading for an observation afterwards is taken only in one position, or with the face of the limb one way, either to the right or left.

It will be proper now to describe the vernier, which serves to read accurately smaller divisions of the limb than could be done with a simple index. The vernier is a short arc of a circle, in which divisions are cut, as in the limb, but smaller; so little smaller however, that the difference between them shall be equal to the smallest denomination which the instrument is intended to read. If the first division of the vernier, marked with a crow-foot, and called the zero of the vernier, be made to coincide with a division of the limb, the last division of the vernier will be observed also to coincide with a division of the limb; and, on counting the divisions, there will be found to be one more on the vernier than upon the same length of the limb. Suppose the smallest division of the limb to be denoted by a, and a division of the vernier by x, and suppose that n' divisions of the vernier are equal to n of the limb, then

$$n'x = na \qquad (1)$$

$$x = \frac{n}{n'}\,a \qquad (2)$$

$$a - x = a\left(1 - \frac{n}{n'}\right) \qquad (3)$$

This last being the difference between a division on the vernier, and a division on the limb, is the expression for the smallest denomination which can be measured with the instrument.

A common division of the limb of the theodolite is into degrees or half degrees, or 30′ spaces; and of the vernier such that 30 of its divisions cover 29 of the limb. The second member of formulas (3) above, by the substitution of these numbers, becomes

$$30'\left(1 - \frac{29}{30}\right) \text{ or } 1'$$

The smallest angular space which the instrument is capable of measuring in this case is one minute of a degree.

The first dividing line or zero of the vernier coinciding with a dividing line of the limb, if the vernier be moved forward, till the second dividing line of the vernier coincides with the next line of the limb, the zero of the vernier will have moved past the dividing line of the limb, at which it stood a distance equal to the difference of a space on the vernier, and a space on the limb, or once the smallest denomination which the instrument measures. If the vernier be moved still farther forward, till its third dividing line, or second from the zero, coincides with the line of the limb, the zero of the vernier will have moved forward again, a distance equal to the difference between a space on the vernier, and a space on the limb, or altogether twice the smallest measure of the instrument. To take a reading, therefore, from a theodolite divided as above mentioned, observe the number of degrees and half degrees pointed out on the limb by the crow-foot or zero of the vernier, and if this line be a little past one of the dividing lines of the limb, count the number of divisions on the vernier from the zero to a dividing line which coincides exactly with one of the limb; this will be the number of single minutes to be added to the degrees and half degrees indicated by the crow foot.

The use of two verniers 180° apart, in correcting for excentricity, may be thus explained. An excentric angle, or one having its vertex not in the centre of a circle, is measured by half the sum of the opposite arcs. The mean of the two opposite verniers, therefore, gives the true angle moved over by the telescope. If the number of verniers be increased beyond two, error of graduation and of figure in the circle are proximately eliminated, as well as error of excentricity, by taking the mean of all the verniers.

TRIANGULATION OF A COUNTRY.

This is a process which consists in measuring a base, and taking the angles at its extremities with the theodolite, for the purpose of determining the positions of points, the sides of the triangles thus determined becoming bases for new triangles.*

SURVEY OF A LARGE ESTATE.

A good instrument for the purpose is a compass with telescopic sights, graduated from 0 to 360°. And a good method of proceeding is to select three stations on elevated ground at remote parts of the estate. Setting out from one of them, go in the direction of one of the others, having previously taken the bearing of the line joining them with the compass. This line may be kept by its bearing in plunging into the low grounds, out of sight of the principal stations. Distances must be measured along the line to points opposite objects on either side of it, which are to be introduced upon the map, and offsets to these objects measured. Oblique instead of perpendicular offsets will sometimes be found more convenient, in which case their bearing must be observed with the compass as well as their lengths measured. Or if an object be too inaccessible to measure the offset to it, its bearing with the compass may be taken from two points on the principal line, which will serve to fix its position. The method of keeping the field book is exhibited on the next page but one.

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* This will be fully treated in our chapter on Geodesy.

The record commences at the bottom of the page, and goes upwards, till one page of the field book is filled, and then commences at the bottom of the next, and so on.[*] The bearings and distances are entered in the middle column of the page, the offsets on the right and left. Distances are usually measured in chains and links, or decimals of a chain. The objects to which the offsets are measured may have their names written, or still better, may be roughly drawn. The points at which streams, roads, stone walls, hedges, &c. cross the principal line, are indicated in the field book by drawing representations of these in the proper direction on both sides of the central column, and where they cross obliquely, recording their bearings upon them.

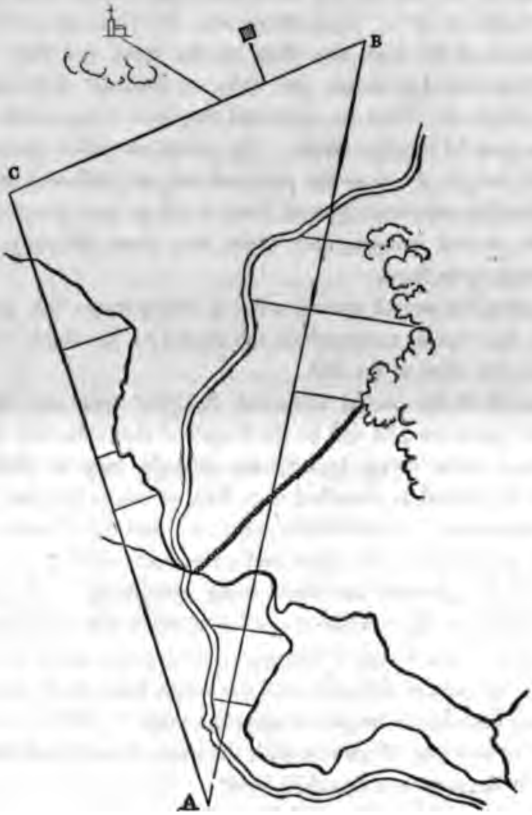
On reaching the second station, a line is drawn across the page, and a line is *run* in a similar manner from the second to the third station, and finally from the third to the first.

The sum of all the partial measured distances upon any one of these sides of the great triangle will be the length of that side, and the lengths of the three sides being known, the triangle may be plotted. The offsets will be plotted as described at p. 243, or where they are oblique, in an obvious manner. A turn to the right, on reaching a station, may be marked ┌ at the side of the page, and a turn to the left ┐.

The great triangle with the offsets being completed, other points may be taken within, as the vertices of smaller triangles having the sides of the large triangle for bases, and, if necessary, other points again within these, until all the systems of triangles with the offsets from their sides include every object desirable to be placed upon the map.

On p. 246 is a map of ground with the great triangle and offsets, and on p. 247 is the corresponding field book.

[*] This is in order that the book and the ground may lie in corresponding positions before the eye of the surveyor.


B
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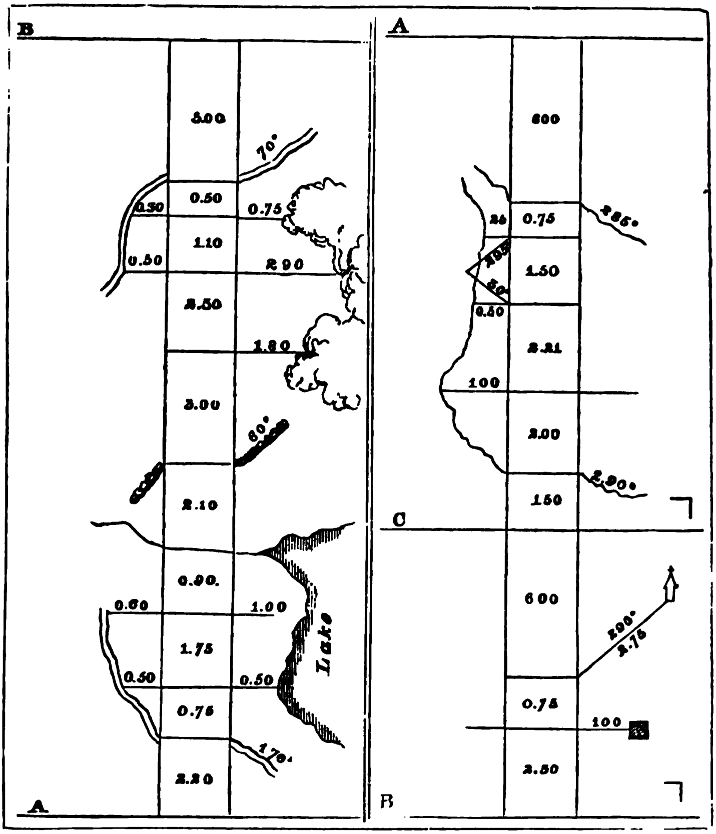

To survey a town or city the same general method would be pursued. The principal lines run with the compass would be the main streets, and the objects on either side would be brought in by either oblique or perpendicular offsets. The points of crossing of the minor streets would be noted, and their bearings taken.

LEVELLING.

The instrument employed for this purpose is called *the level*, and consists of a telescope mounted horizontally upon a tripod. At the top of the tripod, firmly fastened to it, is a small horizontal circular plate of brass, parallel to which, and a few inches above it, is another of the same size, the two being separated by levelling screws, and called levelling plates. From the centre of the upper levelling plate rises vertically a spindle, which fits into a socket in the middle of a horizontal bar, about a foot or more in length. At the extremities of this bar are two stout uprights of two or three inches in length, the tops of which are formed as Y^s, in which the telescope rests, in a position parallel to the horizontal bar. One of these Y^s has a vertical motion, by means of a screw underneath it, which passes up through the end of the horizontal bar. A spirit level is suspended below the telescope and parallel to it, having a horizontal movement by means of a screw at one end, and a vertical movement by the same means, at the other.

1. To collimate the instrument, bring the intersection of the wires upon some well-defined distant point, and then turn the telescope on its optical axis in the Y^s, till the spirit level comes at top. If the intersection of the spider lines be not in the axis of motion, it will depart from the object, and will be as much on the opposite side of the axis after this demi-revolution. It must, therefore, be brought back half way to the object, by the screws which move the wires. This experiment must be repeated till the intersection remains on the object, during the revolution of the telescope on its optical axis.

There are three lines of the instrument which ought to be parallel to each other, horizontal, and perpendicular to the vertical axis about which the instrument turns, viz., the line of collimation, the axis of the spirit level, and the horizontal bar.

2. To render the axis of the spirit level parallel to the line of collimation, bring the air bubble to the centre by turning the levelling screws, having previously placed the level in the direction of the line joining two of them; take the telescope out of the Y^s, and reverse it, turning it end for end; if the bubble remains in the centre, both lines, that is to say, the line of collimation and the axis of the level, are horizontal; if not, bring the bubble half way back to the centre, by means of the screws at the end of the spirit level, and the other half by the levelling screws. Repeat this process until, in both positions of the telescope, the bubble remains in the centre. The axis of the spirit level may also be oblique to the line of collimation in a lateral direction. To ascertain whether it is so or not, turn the telescope on its optical axis as it rests in the Y^s, till the spirit level comes out at one side, not so far, however, as to cause the bubble to disappear; the lateral obliquity will then be converted into an obliquity partially vertical, which the departure of the bubble from the centre will render sensible. This obliquity must be corrected by the screws at the other end of the spirit level, which give a lateral motion to its tube.

3. To render the line of collimation and axis of the spirit level now

parallel to each other, both parallel to the horizontal bar, or rather perpendicular to the vertical axis of the instrument, round which the horizontal bar turns, by means of the spindle fitting into the socket at its centre; bring the bubble to the centre, by means of the levelling screws, turn the instrument round the vertical axis 180°,* and if the bubble remains in the centre, the axis of the spirit level, and consequently its parallel, the axis† about which the telescope revolves, are perpendicular to the vertical axis of the instrument, and the latter, though not vertical, is in a vertical plane, perpendicular to the line of collimation. To make it vertical, which must be done every time an observation is made with the instrument, the above adjustments being completed, bring the bubble to the centre by means of the levelling screws, then turn the bar 90° on the vertical axis, and bring the bubble to the centre again; the vertical axis will then be truly vertical.

In connexion with this instrument two rectangular staves, called levelling staves, are used, divided into feet, tenths and hundredths of a foot. The staves are painted white, and the division lines are made very distinct, in black or red, the feet being numbered with distinct numerals; the hundredths are painted, the whole of each division, alternately white, and black or red. Two or three lengths of 6 feet of staff fit together by joints, so as to make a length of 12 or 18 feet. A plumb line suspended at the side of the staff, or in a groove covered, the lead part with glass so as to be visible, serves to place the staff exactly vertical.

In order to find the difference of level between two points of ground or the height of one above the other with these instruments, let an assistant hold one of the levelling staves vertically, standing upon one of the points in question, and another assistant the other staff, at some point in the direction of the second point; the observer turns the telescope, after the vertical axis of the instrument has been made truly vertical, and takes the reading cut by the horizontal wire, first on one of the levelling staffs, and then on the other,‡ recording them in separate columns of a field the one in he direction in which he is going, under the title of "forward readings," the other under that of reverse readings.* The staff on the given point of ground is now taken up and carried round the other (which is simply turned about where it stands), to a point still farther in the direction of the second given point; the level is also removed to a point between the levelling staves, and the process just described is repeated, and so on, till the second given point, between which and the first the difference of level is required, is reached, the last levelling f being placed there. The difference between the sum of the direct readings and the sum of the reverse readings will be the difference of level of the two given points, or height of the one above the other.

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* If there is a horizontal compass box attached to the instrument this may be done by observing the number of degrees to which the needle points. If not, the instrument must be placed on the line of two staves, and between them, sighting first to one, and then to the other.

† This axis, by the first adjustment, is identical with the line of collimation, or line of vision, when the intersection of the wires is brought upon an object. It is not the optical axis of the telescope, though it should be as nearly as possible, but the axes of the cylindrical parts of its tube, which are in contact with the Y's.

‡ Not only feet, tenths, and hundredths may be distinctly read, but even thousandths, by estimation.

To make a section of ground which is the intersection of a vertical plane with the surface of the earth, it is necessary to add three more columns to the field book, a column of differences, a column of heights, and a column of distances. The first contains the difference between each t and reverse reading;† the second contains the height of each point which the levelling staves are placed above a horizontal plane, med at pleasure, usually passing through the first point of the survey. number in the column of heights will be computed by adding the ber of the column of differences to the preceding number in the column of heights; the first number in the column of heights is the

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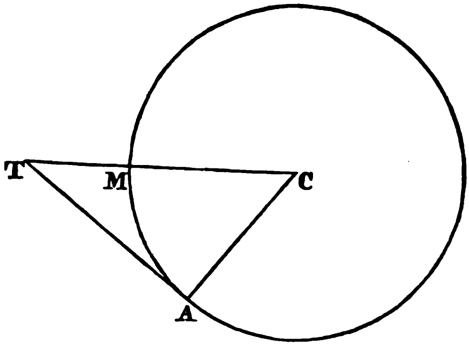
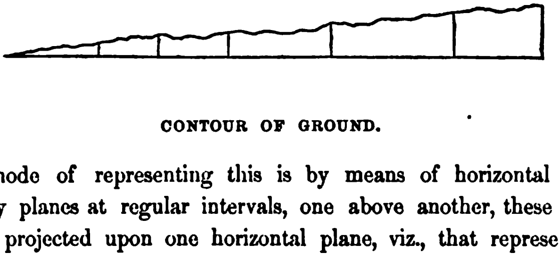
* The level is always placed nearly half way between the two staffs, to avoid the error which would be occasioned by the difference between true and apparent level, the nature of which may be explained by the annexed diagram. Let the circle in the diagram be a vertical section of the earth through the point A, where the instrument is supposed to stand. The arc AM will be the line of true level, A and M being at equal distances from the centre of the earth C; the line AT tangent at A is the line of apparent level, determined by the optical axis of the instrument. For a point on the opposite side of A, from M, and at the same distance, the difference between apparent and true level will be the same. Hence the advantage of placing the instrument half way between the points whose difference of level is to be observed.



† This should be entered with the negative sign, if the direct exceed the reverse reading.

same as that adjoining in the column of differences. The numbers in the column of heights will be negative for points below the plane of reference. The third column contains the horizontal distances between points at which the levelling staves were placed, supposed to be measured with a tape or chain. To plot the section or profile of ground from such a field book, a horizontal line must be drawn to represent the section of the profile with the plane of reference; on this the numbers from the column of distances must be laid off from a scale of equal parts, and at the points of division ordinates or perpendiculars must be erected and taken from the scale of equal parts, equal to the numbers in the columns of heights; through the tops of these ordinates the section of ground required can then be traced with the hand. The following example of a field book, and the profile constructed from it, will serve to illustrate this subject.

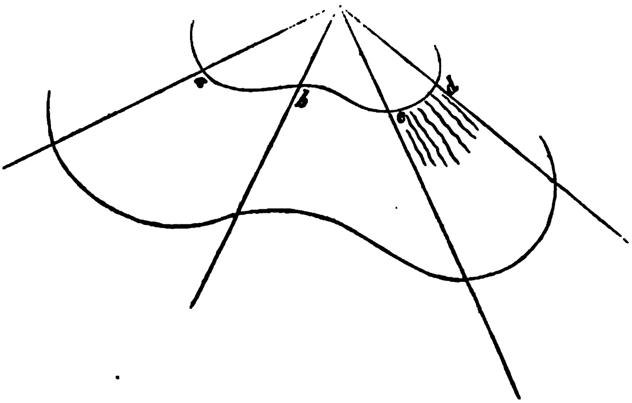
D R	R R	Dif.	H	Dist.
6·895	2·461	4·434	4·434	1·35
5·321	1·468	3·854	8·287	0·75
8·264	3·812	4·452	12·739	1·00
9·322	2·111	7·211	19·950	1·50
7·444	3·212	4·232	24·280	2·00
4·321	1·211	3·110	27·390	1·25



CONTOUR OF GROUND.

The mode of representing this is by means of horizontal sections formed by planes at regular intervals, one above another, these sections being all projected upon one horizontal plane, viz., that represented by the map.

To survey a hill for the purpose of drawing its horizontal sections upon a map, place an instrument for taking horizontal angles at the top of the hill, and plant stakes along lines diverging from this point down the hill, and take the horizontal angles formed by the vertical planes in which these lines lie; then level along the lines, taking the difference of level and distance at points where the slope changes. In order now from such a survey to plot the horizontal sections of the hill, draw on the map from the point A, corresponding to the one assumed on the top of the hill (supposed to have been previously determined in position on the map by triangulation or otherwise), a system of lines diverging from this point under



angles equal to the observed angles, and supposing, to render the conception definite, that the planes forming the horizontal sections are taken 10 feet apart, the points in which the projections of the sections cut the diverging line thus drawn, may be ascertained by proportion as follows:— The difference of level between two points on one of these lines is to the horizontal distance between them, as 10 feet is to the horizontal distance from the upper point, at which a plane of section ten feet below this point would cut the diverging line. Commencing with the upper point of all, A, and determining thus the points *a*, *b*, *c*, *d*, at which the projection of the section 10 feet below cuts each of the diverging lines, it may be traced through them by the hand. In a similar manner another section 10 feet lower, and so on to the bottom of the hill. Where the sections are convex, are the ridges, or *back bones* of the hill; where the sections are re-entering are the ravines; and in general, the varying forms of the sections present to a practised eye an exact notion of the general configuration of the hill. In topographical maps these sections are drawn in pencil, and the hill shaded with irregular lines, in India ink, perpendicular to the sections, as seen between *c* and *d*, which is the direction in which water or alluvion would flow down the hill. These shading lines are best drawn with a pen of short coarse nib, and very short portions of a number of them at a time. The light should be supposed to fall obliquely in a certain direction, and the sides of the ravines towards the source of light will be darker than the opposite sides. The tops of the back bones will be quite light. The summits of the hills being usually more nearly perpendicular, will present deeper shades than the bases, which usually slope gently into the level ground.

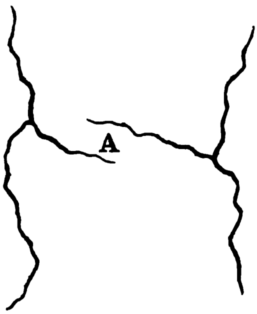
Water is shaded by making the line of shore quite black, which should be first drawn all round the map, including the islands, then a second and finer line all round, as near the first as possible without touching it, then a third a little more distant, and so on, broader and broader, till the shadings from opposite shores meet midway.*

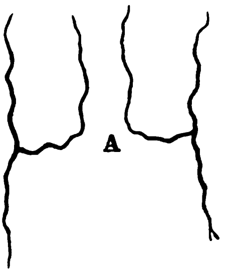
SURVEY OF ROADS, RAILWAYS, AND CANALS.

The problem ordinarily is to ascertain the shortest and cheapest route from one point to another on the surface of the earth, having no greater and more sudden elevations or depressions than are compatible with the nature of the locomotion to be employed upon it.†

The case frequently presented, not only for the whole route, but in minor instances, along its course, is that of a dividing ridge between two valleys, from one to the other of which it is necessary to pass.

It becomes necessary then to ascertain the lowest point of the dividing ridge. A good indication may be obtained proximately by a simple inspection of the course of the streams upon a map, inasmuch as the direction of their flow must be governed by the topography of the country over which they pass. One or two examples will sufficiently illustrate the principles which are to guide such an inspection.





If two streams run along two valleys, and tributaries proceed from near the same point A of the dividing ridge to empty into them, then the point A is a low point of the dividing ridge. For the waters accumulating for the formation of the sources of the tributaries at the point A must flow from higher ground on both sides of this point.

Again, if two tributaries running at first nearly parallel to the principal streams, turn outward at a point A, this is a low point. For the tributaries at first descending into a lower country than that at their sources, encounter rising ground, which prevents their progress in that direction, and turns them off towards their principals.

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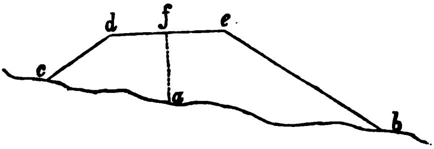
* For further instruction in topographical drawing see an excellent little work by S. EASTMAN, U.S.A.

† This will of course be modified by various circumstances, such as the greater or less difficulty of working the ground, from the nature of the soil; the vicinity of large towns, to pass through which the construction would turn aside, the existence of mines, or valuable products of any kind, for which it would afford transportation.

These examples will suffice to point out the nature of the investigation to which an ordinary map should be subjected. A good topographical map, exhibiting contour of ground, would of course be a far better guide. Conflicting points of passage of a ridge may be compared by means of an altitude and azimuth instrument or theodolite; by first levelling the instrument, elevating the telescope till the line of sight passes through one of the points in question, and then turning the instrument in azimuth till the line of sight passes by the other point in question; if it passes above, the latter point is lower than the former, and vice versâ.

Next, a personal reconnoissance of the country should be made, and information sought from the inhabitants as to the nature of the ground, convenience for obtaining constructing materials, the mineral and agricultural wealth of the region, etc., etc. Three or four routes may thus be selected, one or other of which is to be finally decided upon by a rough survey of them all. This survey is conducted with the compass and chain, and level. The former instruments furnish a plot of the route by the method pointed out at p. 238, and the latter, a continuous profile or longitudinal section of the ground along the whole route, as seen at p. 251. A comparison of the compass plots of the different routes will determine which is the shortest, in a horizontal direction, and a comparison of the profiles will show which presents most elevation and depression to be overcome. The route being thus finally selected, it must be surveyed with care, another column being added to the field book, entitled *grade*, the numbers in which express the height of the roadway or bottom of the canal above the plane of reference at the same points of the route, for which the column of heights expresses those of the natural ground. The numbers in the column of grade will depend upon the elevation or depression of the natural ground, and the slope which the construction permits, that of a common road being greater than that of a railroad, and the latter being greater than that of a canal. The determination of these numbers will require an exercise of eye and judgment. A prime object to be had in view is the equalization of excavation and embankment; *i. e.*, the grade should be so adjusted to the natural slope of the ground as to cut off as much earth as would be required to fill the depression adjoining, up to the level of the grade. The survey being finished, a double profile must be made, one of the natural ground in black ink, the other of the grade in red, upon the same base line, and with the same abscissas or horizontal distances between the ordinates or lines of heights. An inspection of this, and a computation, if necessary, of areas between the black and red lines of section, will serve to show how well the excavation and embankment have been equalized, and the result may require a modification of the grade to adapt it better for the purpose in question, to the natural ground; cross-sections of the route must also be surveyed at all points of change in the longitudinal or latitudinal slope, and more frequently, if these occur at long intervals. The amount of excavation and embankment may be then obtained with sufficient accuracy for an estimate of expense, by computing the areas of the cross-sections of the work as it will be when completed, and multiplying half the sum of the areas of two of them by the longitudinal distance between them. The cross-sections, when the work is in embankment, will be of the form exhibited below, and the same turned upside down when the work is in excavation. These are easily drawn from the field book. The cross-section *bac* being plotted, *a* corresponding to the point where it intersects the longitudinal section of ground, *af* will be the difference of the numbers in the columns of height and grade, *df*, *fe*, each equal to half the breadth fixed upon for the top of the road or bottom of the canal, *dc*, *eb* are then drawn at the proper slopes, for common earth 1½ base to 1 in height. These sections may be regarded as quadrilaterals, and each divided into two triangles, for the purpose of obtaining their area.* Where the cross-section of ground is parallel to the top of the road, or bottom of the canal, they are trapezoids, and the area will then be ½ the sum of the



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* To compute the area of the cross-section from the numbers in the field book, conceive parallels to be drawn from the points *b* and *c* in the diagram, to *af*, meeting *de*, produced both ways; a trapezoid will be formed, from which, if two right angled triangles be deducted, the area of the sections will be obtained. If *d* denote the difference of level between *b* and *f*, *d* will be one parallel side of the trapezoid, and the altitude of one of the triangles, the base of which if the slope be 1½ will be $\frac{3}{2}d$. And if d' denote the difference of level between *c* and *f*, similar expressions will be had for the other side and altitude; and the expressions for the area of the section will be, *b* being the breadth of the roadway,

$$\tfrac{1}{2}(d+d')\left[b+\tfrac{3}{2}(d+d')\right]-\tfrac{3}{2}d\times\tfrac{1}{2}d-\tfrac{3}{2}d'\times\tfrac{1}{2}d'=\tfrac{1}{2}\left[(d+d')b+3dd'\right]$$

parallel sides *de*, *cb*, multiplied by the altitude *af*. Notes of the nature of the soil are kept in the field book of the compass, and the amount of digging and wheeling or carting that can be done in a day in different kinds of earth, having been ascertained by experiment, and the price of day labor being known, the data for determining the expense of the work are all known. To the above must be added the accidental expenses of culverts, blasting rocks, construction of tunnels, &c., which are all subjected to the same general rules, and are functions of the price of materials, mechanic labor, and experiments as to relations of time and amount of performance.

TO COMPUTE THE CONTENTS OF FIELDS.

1. Compute the contents of the figures, whether triangles or trapeziums, &c., by the proper rules for the several figures. If the linear measures be in links, the result is acres, after cutting off five figures on the right for decimals. Then bring these decimals to roods and perches, by multiplying first by 4, and then by 40.

2. In small and separate pieces, it is usual to cast up their contents from the measures of the lines taken in surveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, measured by offsets, all the parts between the offsets are most accurately measured separately as small trapezoids.

4. But in larger pieces, and whole estates, consisting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the measures of the lines and angles that were taken in surveying. For, then, new lines are drawn in the fields in the plan, so as to divide them into trapeziums and triangles, the bases and perpendiculars of which are measured on the plan by means of the scale from which it was drawn, and so multiplied together for the contents. In this way the work is very expeditiously done, and sufficiently correct; for such dimensions are taken as afford the most easy method of calculation; and, among a number of parts thus taken and applied to a scale, it is likely that some of the parts will be taken a small matter too little, and others too great; so that they will, upon the whole, in all probability, very nearly balance one another. After all the fields and particular parts are thus computed separately, and added all together into one sum, calculate the whole estate independently of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add these also together. Then if this sum be equal to the former, or nearly so, the

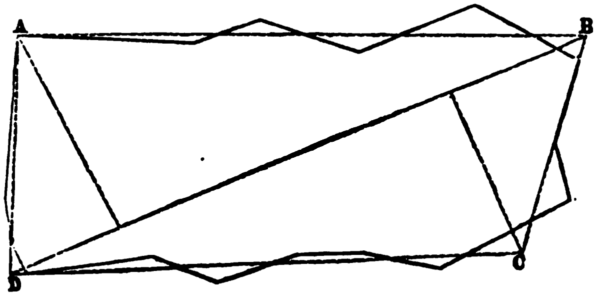
work is right; but if the sums have any considerable difference, it is wrong, and they must be examined and recomputed, till they nearly agree.

5. But the chief secret in computing consists in finding the contents of pieces bounded by curved or very irregular lines, or in reducing such crooked sides of fields or boundaries to straight lines, that shall inclose the same or equal area with those crooked sides, and so obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now, this reducing the crooked sides to straight ones, is very easily and accurately performed in this manner:—Apply the straight edge of a thin, clear piece of lantern-horn to the crooked line which is to be reduced, in such a manner that the small parts cut off from the crooked figure by it, may be equal to those which are taken in; which equality of the parts included and excluded you will presently be able to judge of very nicely by a little practice; then with a pencil or point of a tracer, draw a line by the straight edge of the horn. Do the same by the other side of the field or figure. So shall you have a straight-sided figure equal to the curved one, the content of which, being computed as before directed, will be the content of the curved figure proposed.

Or, instead of the straight edge of the horn, a horse-hair may be applied across the crooked sides in the same manner; and the easiest way of using the hair is to string a small slender bow with it, either of wire, or cane, or whalebone, or such like slender or elastic matter; for, the bow keeping it always stretched, it can be easily and neatly applied with one hand, while the other is at liberty to make two marks by the side of it, to draw the straight line by.

EXAMPLE.

Let it be required to find the contents of the irregular figure below, to a scale of 4 chains to an inch.



Draw the four dotted straight lines AB, BC, CD, DA, cutting off equal quantities on both sides of them, which they do as near as the eye can judge; so is the crooked figure reduced to an equivalent right-lined one of four sides, ABCD. Then draw the diagonal BD, which, by applying a proper scale to it, measures 1256. Also the perpendicular, or nearest distance, from A to this diagonal measures 456; and the distance of C from it is 428.

Then, half the sum of 456 and 428, multiplied by the diagonal 1256, gives 555,152 square links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

TO FIND THE CONTENT OF A FIELD WITHOUT PLOTTING.

Take the bearings and lengths of the sides of the field, and enter them in a field book as course and distance, and take out the difference of latitude and departure corresponding to each, and enter them in two double columns, marked N. S. and E. W. as at Art. 98. To obtain these, if the bearings are given in degrees, recourse may be had to a table of difference of latitude and departure for every degree and minute of the quadrant, such as is found in Bowditch's Navigator, or instead of this, the difference of latitude and departure may be calculated for each course and distance.* Another double column must be added, entitled double meridian distances. The meridian distance of any line is the distance of its middle point from an assumed meridian, which should be taken through some corner of the field. The double *meridian distance*, corresponding to *the first course* adjoining the assumed meridian, will be equal to *the departure of that course.* Double the meridian distance of any other course will be equal to the *double meridian distance of the preceding course, plus the departure of the preceding course, plus its own departure.*†

In applying this rule, distances to the right should be considered +, those to the left —. The double meridian distances east of the meridian should be entered in a column marked E., and those west in a column marked W., the column of double meridian distances being made double for the purpose. By means of this double column of double meridian distances, and the double column of differences of latitude, the content of the field may be computed by the following rule.

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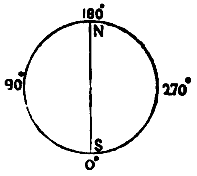
* The sum of the numbers in the column marked N. ought to equal that of the numbers in the column marked S. If such be not the case, the difference between the two sums should be half of it subtracted from the numbers in the column having the greater sum, being distributed among them in proportion to their magnitude; the other half should be added in the same way to the numbers in the column producing the less sum. For in going round a field and returning to the same point, the distance gone north must be equal to that gone south. The same remark applies to the columns marked E. and W. New columns will then be derived which may be called corrected diff. of lat. and departure.

† This may be seen by making and inspecting a diagram.

The difference between the northings multiplied by the eastings plus the southings multiplied by the westings, and the northings multiplied by the westings, plus the southings multiplied by the eastings, will be equal to double the area of the land.

The proof of this is left as an exercise for the student.

In the following example the bearings were taken with a compass resembling the mariner's in principle. A metallic graduated circle, one diameter of which, that joining the zero and 180° points, being an attached needle, the graduated circle was held stationary in space by the magnetic force of the earth. The numbering was from zero to 360 in the direction shown in the annexed diagram. The compass sights were plain, and the number on the line of sights towards the extremity next the eye was the one read and recorded in the 1st column p. 261. The equivalents of these readings in bearings of the compass courses or sides of the field from the meridian, are recorded in the 2d column. These are ascertained by considering in what part of the circumference in the diagram above the No. in the 1st column would fall; the course would be in the direction from this point to the centre of the circle.



The third column contains the lengths of the courses or sides of the field, measured with a chain. Then follow the columns of difference of latitude and departure,* and the columns of corrected diff. of lat. and departure. The assumed meridian from which to estimate the double meridian distances is taken through the point at which the survey commenced.† The double meridian distance of the first course then, according to the rule, will be equal to its departure 234·1, and is W. because the departure is W. Double the meridian distance of the second course is equal to that of the preceding course 234·1 + the departure of the preceding course 234·1 + its own departure 50·3. (See rule.) All these numbers being W., their sum in the arithmetical sense, 518·5, is taken as the double meridian distance of the 2d course. For the next course the departure 17·6 is E., and on the general analytic principle that quantities estimated in a contrary sense must have contrary signs, this may be ered f we regard the previous numbers employed in computing the ouble meridian distances, which are all W. as negative. To add this 17·6 then, in e algebraic sense, to the sum of 518·5, and 50·3, according to the rule, will be in i which gives 551·2 for the 3d D. M. D. For a similar reason the 2 being both E., must be subtracted from 551·2, which is W., to roduce M. D.; and so on till we arrive at the 7th course, marked 80°, in the st colu the sum of the two departures, 96·7 and 59·2, both E., viz., 155·9 exceeds the last D. M. D., 135·9, which is still W., and as they have contrary signs, their algebraic sum will be their difference with the sign of the greater, which is E., and this difference 20·0 must be entered in the column E. of double meridian distances. The D. M. D. of the last course is obtained by adding 59·2 and 20·0 both ., and subtracting 40·1, which is W., from their sum. The double meridian distances E., which have corresponding difference of latitude, N., are now multiplied by m according to the rule; and the double meridian distances W., which have corponding differences of latitude S., and the products all entered in a column entitled N. × E. + S. × W., and their sum taken. Of the N. × W. + S. × E., which he remaining part of the rule requires to be formed, there is but one product in this ample an N × W. 196·2 by 234·1, or 45930·42, for which an additional column, hich would ordinarily be employed, is not worth while. This product is subtracted rom the sum of the former, and the remainder, 68103·90, is by the rule equal to double the area of the land in square links, 10,000* of which make a square chain. lf this will be the area, which is converted into square chains by removing the imal point 4 places to the left; and this again into acres, by removing the decimal point one place further to the left still, since there are 10 sq. chains in an acre. The decimals of an acre are converted into roods and perches by multiplying by 4 and by 40.

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* The sum of the column N. 264·5 exceeds that of the column S. 263·0 by 1·5; half this, or ·8 is subtracted from the numbers of the first column N., ·6 from the 1st No. in the column, and ·1 from each of the other two, to obtain the Nos. in the 2d column N., and ·7, in the same manner, is added by distribution among the five numbers of the 1st column S., to produce those of the 2d column S., &c.

† It would be most simple to assume it through the westernmost point of the land, in the present example at the commencement of the 3d course, where the reading was 163°. Here the courses, which were previously all W., begin to turn E. The advantage of this is that the double meridian distances would be all E.

The plotting of the above example will be an exercise. A circle should be described on the paper, and points marked on it, according to the compass readings in the first column, the N. and S. line corresponding to the 0° and 180° points, as in the last diagram. Lines drawn from the points thus marked to the centre of the circle will be parallel to the boundary lines of the survey. For further directions see p. 237 at bottom.

* Which is the square of 100, the No. of links in a chain.

EXAMPLE.

Compass Reading.	Bearing from Meridian.	Dist.	Diff. Lat.		Dep.		Cor. Diff. Lat.		Cor. Dep.		D. M. D.		
			N.	S.	E.	W.	N.	S.	E.	W.	E.	W.	N.×E.+S.×W.
310°	N. 50 W.	306	196·8			234·4	196·2			234·1		234·1	
226	S. 46 W.	70		48·6		50·4		48·7		50·3		518·5	25250·95
163	S. 17 E.	60		57·4	17·5			57·5	17·6			551·2	31694·00
133	S. 47 E.	100		68·2	73·1			68·3	73·2			461·4	31513·62
129	S. 51 E.	100		62·9	77·7			63·1	77·8			310·4	19586·24
105	S. 75 E.	100		25·9	96·6			26·1	96·7			135·9	3546·99
80	N. 80 E.	60	10·4		59·1		10·3		59·2		20·0		206·00
325	N. 35 W.	70	57·3			40·2	57·2			40·1	39·1		2236·52
			264·5	263·0	324·0	325·0	263·7	263·7	324·5	324·5			114034·32

45930·42

2)68103·90

Area in Square Links, 34051·95
4

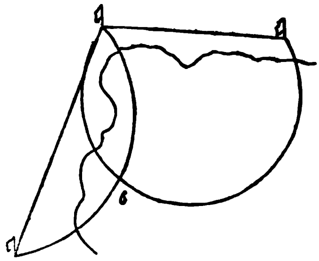
Roods, 1·3620780
40

Perches, 14·4831200

The area is therefore 1 rood, 14·48 perches.

HYDROGRAPHIC SURVEYING.

In the survey of harbors, after having surveyed and plotted the outline of the shore, it becomes necessary to set down upon the map the depths of the water in feet or fathoms at a sufficient number of points to serve as a guide to navigators. The depth is ascertained by sounding, and the problem is to fix upon the map the points at which the soundings were made. One method consists in rowing a boat uniformly in a straight line from one point on the shore to another opposite, casting the lead at regular intervals by a watch; this line being drawn on the map and divided into as many equal parts as there were casts, the points of division will be the points required; upon these the numbers obtained by the soundings are to be put down. Another method is to place three signals upon the shore, not in the same straight line, and with a sextant* in the boat to measure the angles subtended by the lines joining these signals; then having these lines plotted upon the map, construct upon each of them a segment capable of containing the observed angle subtended by it (see Plane Geom., Prob. 21), and the intersection of the arcs of these segments will determine the points on the map at which the boat was situated at the time of observation. The sounding of course should be taken at the same point, and recorded at its position (6) thus determined on the map.



A third method consists in having two theodolites, and taking the angles with them from the extremities of a base line on the shore, by which means the position of the boat is determined. In this method a system of signals is requisite, by which the observers on shore may know the instant at which the sounding is made.

A very perfect one was invented by Mr. Thomas H. Norris, of New York, and practised in the survey of the mouths of the Mississippi. This consisted in having at one of the two stations on the shore (which in the low lands of the Mississippi were elevated platforms of wood) a flag which could be run up and a chronometer. The boat also carried a chronometer. The intervals of time at which the soundings should be made having been previously agreed upon, about 10 seconds before one of these intervals expired, the flag was run up, and both theodolites brought to bear upon the boat, or rather upon a staff at its bow, from which the sounding was made. The tangent screws served to keep the instruments steadily upon this point, and the instant the 10 seconds were up, the flag was lowered, the lead was cast, and the readings taken from the horizontal limbs of the instruments and recorded. This mode was found to be very rapid and accurate. By way of experiment the boat was frequently made to cross its track, and the agreement was exact. By rowing the boat along at oar's length from the shore, and determining its position at frequent intervals, as above described, the line of shore could be traced upon the map. This was found particularly convenient in the survey of the bayous or inlets of the low muddy banks.

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* See the instrument of this name described at p. 290, note.

Horizontal sections of the bottom of a harbor may be determined, making the plane of the water a plane of reference, in an obvious manner, and the bottom represented in the same manner as a hill. This, however, is not often practised.

Points at great distances out at sea are obtained by triangulating outward with three vessels successively moored at points more and more remote from the shore.

PART V.

APPLICATION OF SPHERICAL TRIGONOMETRY TO

NAUTICAL ASTRONOMY.

[illegible]

[illegible]

PART V.

APPLICATION OF SPHERICAL TRIGONOMETRY TO

NAUTICAL ASTRONOMY.

103. NAVIGATION, as we have seen, is the determination of the place of a ship at sea, that is to say her latitude and longitude, by the "dead reckoning."

The dead reckoning proceeds upon the hypothesis that the ship's course and the distance she sails are accurately known; and if this were really the case, her true place might be found by the methods given in Part III. But this is impossible. 1. From the difficulty of steering exactly upon the intended course. 2. From the uncertainty of lee-way. 3. From errors of the log, occasioned by the heaving of the sea, unknown currents, and the rudeness of the instrument itself.

The "dead reckoning" is, however, indispensable in determining the ship's place during cloudy weather, and is useful at all times for detecting the existence and velocity of currents.

The main reliance must be upon astronomical observations, and the method of determining a ship's place by means of these constitutes the science of nautical astronomy.

DEFINITIONS.

104. For the purpose of measuring the *angular* distances of the heavenly bodies from each other, and from the horizon, it is convenient to suppose them all situated as they really appear to an observer on the earth, viz., in a spherical concave surrounding the earth, and concentric with it. This imaginary concave, which the student may suppose identical with the blue vault of the sky, is called the celestial sphere.

The position of a point on the celestial sphere, like the position of a point on the terrestrial sphere, is fixed by its latitude and longitude. On the celestial sphere the circle of longitude is the ecliptic; and secondaries passing, therefore, through the poles of the ecliptic, are the circles of celestial latitude; the point from which longitude is measured is the vernal equinoctial point. Commencing at this point, the ecliptic is divided into twelve parts called signs; a sign is, therefore, 30°. The twelve signs are named, and symbolically expressed, as follows:

1. ♈ Aries.
2. ♉ Taurus.
3. ♊ Gemini.
4. ♋ Cancer.
5. ♌ Leo.
6. ♍ Virgo.
7. ♎ Libra.
8. ♏ Scorpio.
9. ♐ Sagittarius.
10. ♑ Capricornus.
11. ♒ Aquarius.
12. ♓ Pisces.

The vernal equinoctial point is called the first point of Aries. The longitude is measured from this point in one direction, viz., in the order of the signs, or from W. to E.

Parallels of latitude on the terrestrial sphere correspond to parallels of declination on the celestial. Of these, the two which touch the ecliptic in the first points of Cancer and Capricorn, are called the *tropics* of Cancer and of Capricorn. These first points of Cancer and Capricorn[*] are respectively called the summer and winter *solstice;* because for a day or two before and after the sun enters them he appears to be stationary, and the days to be of equal length, so slowly does his declination at those times change, for his motion is obviously very nearly parallel to the equator. The declination circle, through the solstitial points, is called the *solstitial colure*, and that through the equinoctial points the *equinoctial colure.*

Secondaries to the equator, we have said (Art. 79), are called *declination* or *hour circles.*

The declination of a heavenly body is its distance from the equator in degrees, minutes, and seconds, measured on the declination or hour circle which passes through the body.

The right ascension of a heavenly body is the number of degrees and fractions of a degree measured on the equator, between the vernal equinox or first of Aries, and the circle of declination which passes through the body. Another definition of right ascension is the angle at the pole of the equator or of the earth, comprehended between the hour or declination circle through the vernal equinox, and the hour circle through the heavenly body.

Right ascension is now commonly expressed in hours, minutes, and seconds of time, allowing 15° to the hour, 15′ to the minute, and 15″ to the second of time.

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[*] At the first of these points the sun, which up to the time of its arrival there had been moving north, begins to move backwards towards the south; at the second from going south he begins to climb upwards towards the north, whence it appears that the points in question are named in allusion to the habits of the animals after which they are called.

The right ascension is also the difference in the time of transit of the heavenly body and of the first point of Aries over the meridian of any place.

The difference of right ascension of two stars is the difference in their times of meridian transit at the same place. Or it is the angle comprehended between the two hour circles which pass through the stars.

A *catalogue* of stars is a list of them with the right ascension and the declination of each annexed.*

Having described the principal circle and points of the celestial sphere which are considered as permanent, or which do not alter with the situation of the observer on the earth, we come now to describe those which change with his place. The principal of these is the *horizon*, which has been defined already (Art. 79), and *vertical circles*, which are secondaries to the horizon, and on which the altitudes of celestial objects are measured.

These vertical circles all meet in two points diametrically opposite, viz., the poles of the horizon; one of which is directly over the head of the observer, and called his *zenith*, and the opposite one his *nadir*. The ver-

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* The late catalogue of the British Association, the name of which is abbreviated B. A. C., gives the north polar distances (N. P. D.) of the stars instead of their declinations. The north polar distance of a star is its distance from the north pole of the heavens, measured on the circle of declination passing through the star. The right ascension of the star fixes the position of this circle in the heavens, and the north polar distance fixes the place of the star upon the circle, so that its position is completely determined by these two co-ordinates. In the British Catalogue is a column containing the annual variation in R. A., and four columns marked a, b, c, d, at top; also a column containing the annual variation in N. P. D., and four columns marked a', b', c', d'. The numbers whose logarithms are in these columns may be regarded as constant for a period of about ten years. In the Nautical Almanac, on p. XXII. of each month, will be found four columns marked A, B, C, D, at top, containing the logs. of numbers, which vary with the time, or are ephemeral.

To find the R. A. of a star for any given time, take out its R. A. for the epoch of the catalogue, viz., 1850, to which add the product of the annual variation in R. A., by the number of years between the given time and 1850. The result will be the mean R. A. at the beginning of the given year. Take out from the columns a, b, c, d, the logs. opposite the given star, and from the Nautical Almanac, from the columns A, B, C, D, the logs. corresponding to the given date, for which the apparent R. A. is required, and with these logs. compute the following formula:

$$d\alpha = \text{A}a + \text{B}b + \text{C}c + \text{D}d$$

$d\alpha$ being the correction to be applied to the result before found, to obtain the R. A. required. This will be the time at which a star ought to make its meridian transit by the siderial clock. The formula for the correction in declination is

$$d\delta = \text{A}a' + \text{B}b' + \text{C}c' + \text{D}d'$$

tical circle which passes through the east and west points of the horizon is called the *prime vertical;* it necessarily intersects the meridian of the place (which passes through the north and south points) at right angles.

The *azimuth* of a celestial object has been already defined to be an arc of the horizon, comprised between the meridian of the observer and the vertical circle through the object, and hence vertical circles are sometimes called azimuth circles.

The *amplitude* of a celestial object is the arc of the horizon comprised between the east point and the point where the object rises, or between the west point and that where it sets; the one is called the rising amplitude, the other the setting amplitude.

ON THE CORRECTIONS TO BE APPLIED TO THE OBSERVED ALTITUDES OF CELESTIAL OBJECTS.

105. The *true altitude* of a celestial object is always understood to mean its angular distance from the rational horizon of the observer. This is not obtained directly by observation; but is the result of certain corrections applied to the observed altitude.* These we shall now enumerate and explain.

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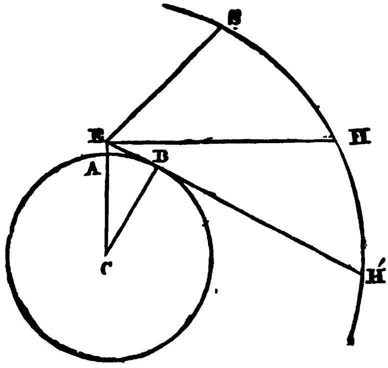
* The observed altitude is obtained by means of an instrument called a quadrant of reflection, or simply a quadrant. This instrument is a frame of wood in the form of a sector of a circle, the arc of which is graduated to degrees and parts of a degree. This frame is suspended so that the plane of the circle shall be vertical. It has an arm, one extremity of which is attached to the centre of the circle, and which is movable about this point; upon this arm is a small mirror, and opposite to it is a plane glass, half of which is mirror, and half transparent. When a heavenly body, seen by double reflection in these two mirrors, is brought by the movement of the arm, upon which one of the mirrors is placed, to coincide with the line of the horizon at sea as seen through the transparent part of the opposite glass, the outer extremity of the arm points out upon the graduated arc the number of degrees of altitude of the heavenly body above the horizon.

The construction of this instrument depends upon the optical principle that the angle of incidence is equal to the angle of reflection. The angular movement of the image of the heavenly body is double the angular movement of the arm, so that to measure the greatest altitudes, the limit of which is 90°, the graduated arc need be but the eighth of a circumference; the degrees upon it are however numbered as if it were a quadrant, to save the trouble of doubling them. The instrument takes its name from the amount which it measures, instead of from the magnitude of its arc. There are colored glasses attached, which can be interposed so that the rays of light, coming from the heavenly body to the eye, can be made to pass through them when taking the altitude of the sun.

More complete instruments of this nature are the sextant and repeating circle, or circle of reflection, for full descriptions of which see p. 290, and p. 299.

DIP OR DEPRESSION OF THE HORIZON.



106. Let E represent the place of the observer's eye, elevated to the height EA above the surface of the earth, and S the place of a heavenly body; the first object is to obtain its apparent altitude above the horizontal line EH; that is, the angle SEH. Now, since to the observer, the *visible* horizon is in the direction EBH′, the altitude taken with the instrument is the angle SEH′; hence from this observed altitude the angle HEH′, called the *Dip* or *Depression of the Horizon*, must be subtracted to obtain the apparent altitude SEH.

The angle HEH′, or its equal C, is calculated for various elevations, AE, of the eye above the surface of the sea, by resolving the right angled triangle EBC, in which are known CB, the radius of the earth, and EC equal to the radius increased by the height of the eye. The results are registered in a table (Table XXXI.), the argument of which is the height of the eye.

The depression thus obtained must be lessened by the amount of terrestrial refraction, which is very uncertain; $\frac{1}{10}$ of the whole quantity has been allowed in computing this table.

SEMIDIAMETER.

107. The foregoing correction for dip having been applied, the result will be the apparent altitude of the object observed, above the sensible horizon. If this be the upper or lower edge of the disc of the sun or moon, called the upper and lower limb, a further correction will be necessary to obtain the apparent altitude of the centre. The angle at the eye of the observer, subtended by the semidiameter or radius of the sun or moon, must be added to the altitude of the lower limb, and subtracted from that of the upper limb. This quantity, which is continually varying both for the sun and moon, in consequence of the variation of their distance from the

is gi he Nautical Almanac for every day in the year.* But in the case of the moon the semidiameter itself requires a small correction depending upon the observed altitude. For the semidiameter, furnished by the Nautical Almanac, is the apparent *horizontal* semidiameter, *i. e.* the apparent semidiameter when the moon is in the horizon, where the distance from the observer is greater than when she is in the zenith by the semidiameter of the earth. Consequently her apparent semidiameter, which is inversely as her distance, will be least in the horion, and greatest in the zenith; and its value between these limits will ary with the sine of the altitude, as may be easily seen by constructing a diagram.

The distance of the moon being about 60 semidiameters of the earth, noon's horizontal semidiameter will be increased about $\frac{1}{60}$ part in the zenith. Therefore, if to the logarithm of the sine of $\frac{1}{60}$ of the ☽'s horizontal semidiameter or the log. of the arc itself, which is small, we add

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It is given for noon of each day for the sun, and for noon and midnight for the n, and is found for any other time of day by the proportion: As 24 or 12 ours : the variation in 24 or 12 hours : : the time after noon or midnight, at Greenwich : the variation in that time, which must be added to the semidiameter given in the Almanac or subtracted, according as the semidiameter is increasing or diminishing from day to day, in order to have the semidiameter at the required time.

Proportions of this kind, in which the terms contain two or three denominations, as hours and minutes, minutes and seconds, or hours, minutes, and seconds, degrees and minutes, &c., may be resolved conveniently by means of the table of proportional logarithms, Table XXII.

The following example will illustrate the mode of proceeding.

$$24^h : 16' \, 19'' : : 8^h \, 2^m$$

Taking the first and third terms one grade lower, we find their proportional logarithms (P. L.) on pp. 134 and 132, writing the arith. comp. of the former, and taking from p. 133 the P. L. of 16′ 19″, the calculation will be as follows:

24^m	ar. comp. P. L.	9·1249
16′ 19″	P. L.	1·0426
$8^m \, 2^s$	P. L.	1·3504
5′ 28″	P. L.	1·5179

In this as in many other problems of Nautical Astronomy, the time at Greenwich at the instant of observation is required, and may be found by adding or subtracting the difference of longitude in time, according as the place is W. or E. of Greenwich. Thus the time at Greenwich, corresponding to any given time at New York, is found by adding $4^h \, 56^m \, 4^s$ (the difference of longitude between the two places) to the latter.

the log. sine of the ☽'s altitude, the result will be the log. of the apparent semidiameter at the given altitude.

In this way is formed the Table at the end, entitled *Augmentation of the Moon's Semidiameter* (Table XXXIII.), which contains the proper correction to be *added* to the given horizontal semidiameter, to obtain the true semidiameter.

On account of the great distance of the sun, no such correction of his semidiameter is necessary.

The corrections for dip and semidiameter being thus applied, the result is called *the apparent altitude of the centre.* In the case of the stars, the only correction for the apparent altitude is the dip.

To obtain the *true* altitude requires two other corrections, viz. for *refraction* and for *parallax.* The former of these has indeed an effect upon the two preceding corrections, dip and semidiameter, which require certain modifications in consequence, which we shall notice after explaining the nature and effect of

REFRACTION.

108. The rays of light coming from a heavenly body, having to pass through the atmosphere, are bent towards the vertical by refraction. As the atmosphere grows more and more dense in approaching the surface of the earth, the light bending continually towards the vertical pursues a curvilinear path in a vertical plane, and enters the eye in the last direction of its motion, which prolonged is a tangent to the curve, and it is in the direction of this tangent that the object emitting the light appears. The curve being convex upward, the tangent lies above it, and the effect of refraction is therefore to elevate the object, or to make the apparent place above the true place. The correction for refraction, therefore, like the correction for dip, is always *subtractive;* it decreases from the horizon, where it is greatest, to the zenith, where it vanishes (as the rays from objects in the zenith enter the atmosphere perpendicularly) in accordance with the optical law that the ratio of the sine of the angle of incidence to the sine of the angle of refraction is constant.

At the end of the volume we have given a table of refractions containing the correction for refraction to be applied to every altitude, from the horizon to the zenith,* and adapted to the mean state of the atmosphere (Table XXX.).* When the temperature of the atmosphere is raised, which is indicated by the thermometer, the refraction decreases; and when the density of the atmosphere is increased (indicated by the rising of the mercury in the barometer), its refractive power increases. The change in refraction for a difference of 1° of Fahrenheit, and of 1 inch in the barometer from the mean state, is given in separate columns, and must be multiplied the one by the number of degrees which the thermometer differs from 50°, and the other by the number of inches and fractions of an inch which the barometer differs from 30°, and the result added or subtracted, as the case may require. It should be observed that below 4° the refraction is very variable and uncertain, and such low altitudes should be avoided as much as possible at sea.

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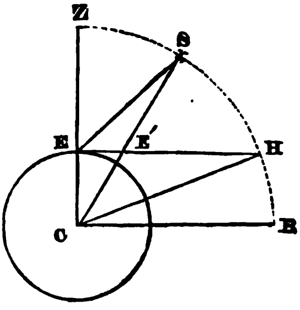
* It will be observed that there is in the table a column of differences for 1′ of altitude. The number in this opposite the degrees in the given altitude must be multiplied by the given minutes and the result subtracted from the correction, or added to the altitude.

It will be unnecessary to use the correction for the state of the barometer and thermometer, when the latitude of the ship is the only object of the observation, as this could seldom make a difference so great as half a mile in the resulting latitude; but, in determining the longitude by the *Lunar Observations*, the neglect of these small corrections would sometimes introduce an error in the resulting longitude of more than thirty miles.

When the foregoing corrections have been applied to the observed altitude, the result will be the true altitude of the centre above the sensible horizon, and it now remains to apply the correction necessary to reduce this to the true altitude of the centre above the rational horizon; that is, to the altitude which the body would have if the observer were situated at the centre of the earth instead of on its surface. This last correction is called

PARALLAX.

109. In order to explain the nature and effect of parallax, let S represent the place of the object observed from the surface of the earth, at E; then the angle SEH, that is, the observed angle, when corrected for dip, semidiameter, and refraction, will be the true altitude of the object, in reference to the observer's sensible horizon EH; and the angle SCR will be the true altitude



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* Such a table might be formed by comparing the observed altitude of a star with its altitude computed from the declination or N.P.D. hour angle and latitude.

in reference to the rational horizon CR; and the difference of these angles is the parallax called *parallax in altitude* when the object is above the horizon as at S, and *horizontal parallax* when it is in the horizon as at H.

Since the angle SE′H is equal to the angle SCR, EH and CR being parallel by definition, we have for the parallax in altitude SE′H — SEH = ESC (Geom. Th. 15), that is, the parallax is the angle which the semidiameter of the earth subtends at the object;* it is obviously greatest in the horizon, and nothing in the zenith, and is the quantity which must be *added* to the true altitude above the sensible horizon to obtain the true altitude above the rational horizon.

The sun's parallax in altitude is given in a Table at the end (Table XXXIV.), his horizontal parallax being nearly constant; and the moon's horizontal parallax is given for the noon and midnight at Greenwich, of every day of the year, in the Nautical Almanac, and from the horizontal parallax thus obtained, parallax in altitude must be calculated. This is easy; for since in the triangle SEC we have the proportion

$$SC : EC :: \sin SEC = \sin SEZ = \cos SEH : \sin ESC;$$

it follows (since SC, the distance of the heavenly body, as well as EC, the semidiameter of the earth, may be regarded as constant for a single day), that the sine of the parallax in altitude varies as the cosine of the altitude; but when the altitude = 0, as in the case of horizontal parallax, cos. altitude = 1, and the constant ratio SC to EC, the above proportion shows to be equal to the sine of the horizontal parallax. But from the proportion itself we see that it is necessary to multiply this ratio by the cosine of the altitude, to have the sine of the parallax in altitude; but as the parallax is always a very small angle, it is usual to substitute the arc for its sine, or par. in alt. = hor. par. × cos. alt., so that

log. hor. par. in seconds + log. cos. alt. — 10 = log. par. in alt. in seconds.

We must observe here that the horizontal parallax, given in the Nautical Almanac, is calculated to the equatorial radius of the earth;

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[*] This result might be arrived at much more simply by means of our definition of an angle (Geometry, def. 10), viz. "the difference of direction of two lines," and a definition of parallax, viz. the difference of direction in which an object is seen from the centre and surface of the earth, or in a more enlarged sense of the term, from any two points. This in the diagram will be the difference of direction of the two lines CS and ES, *i. e.* the angle CSE, or the angle subtended by the line joining the two points of observation.

and, therefore, except at the equator, a small subtractive correction of the horizontal parallax will be necessary, on account of the spheroidal figure of the earth, in consequence of which the radius of the earth is smaller everywhere else than at the equator, and consequently subtends a smaller parallax. A table of such corrections is given at the end. (See Table XXXV.) It must evidently be a table of double entry, the two arguments being the equatorial horizontal parallax and the latitude, upon which two quantities the correction depends.

110. Such are the corrections necessary to be applied to the observed altitudes of celestial objects, in order to obtain their true altitudes. A few other preliminary, but very simple and obvious operations, must also be performed upon the several quantities taken out of the Nautical Almanac, in order to reduce them to their proper value at the time and place of observation; for the elements furnished by the Nautical Almanac are computed for certain stated epochs, and their values for any intermediate epoch must be found by proportion. But ample directions for these preparatory operations are contained in the "Explanation of the Articles in the Nautical Almanac,"* to be found in the last pages of that work.

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* It may be well, however, to give here some general account of the arrangement of the Nautical Almanac. The first twenty-two pages contain the right ascension, declination, semidiameter, and a variety of other elements relating to the sun and moon for every day of the month of January, the right ascension of the sun at mean noon and at apparent noon, that of the moon at the beginning of every hour of mean time throughout the day at Greenwich. The next twenty-two pages contain the same elements for the month of February, and so on, each month occupying twenty-two pages, marked with the Roman numerals, I. II., &c. The year thus being gone through, after a few pages containing the sun's co-ordinates, follows the ephemeris of the planets, beginning with Mercury, the one nearest the sun. This contains the semidiameter and declination, apparent right ascension, as affected by aberration of light, and some other elements of the planet for every day in the year of mean noon at Greenwich, and also at the time of the planet's meridian transit at Greenwich, each month occupying two pages. This Ephemeris extends from p. 275 to p. 455, in the almanac of 1850. The next three pages contain the mean places or right ascension and declination on the 1st of Jan. of 100 principal fixed stars, with their annual variations in right ascension and declination, marked + or —.

The latter multiplied by the fraction of the year which has elapsed, which is given in the last column of p. XXII. of each month, will be the quantity to be added or subtracted, in order to have the mean R. A. and Dec. at the time. To obtain the true places, corrected for nutation, &c., recourse must be had to formulas and tables given in the next three pages of the Almanac, except that of a number of the principal stars, the true R. A. and Dec. are given for every ten days from p. 468 to p. 501, in the edition of 1850. The remaining matters contained in the Nautical Almanac will be noticed as occasion requires.

EXAMPLES OF THE CORRECTIONS.

1. On the 14th of July, 1833, suppose the observed altitude of the sun's lower limb[*] to be 16° 36′ 4″, the observer's eye to be 18 feet above the level of the sea, the barometer to stand at 29 inches, and the thermometer at 58°; required the true altitude of the sun's centre.

Observed alt. ⊙'s L. L.	16°	36′	4″
Depression of the horizon (Tab. XXXI.) . .	—	4	·4
App. alt. of L. L.	16	32	0
Refraction	—	3	14[†]
Correction for Barometer		+	6·5[‡]
Correction for Thermometer		+	3·2[§]
True altitude of L. L. above the visible horizon .	16	28	55·7
Sun's semidiameter (Naut. Alm.)	+	15	45·4
Parallax in altitude		+	8·4[‖]
True altitude of the sun's centre	16	44	49·5

2. On the 23d of June, 1850, in longitude 4^h 56^m 4^s W., latitude about 40° 43′ N., at 11^h 44^m 55^s mean time, the double altitude of the moon's upper limb was observed by reflection from Mercury to be 58° 14′; the index error of the sextant was 15″ subtractive; the barometer stood at 30·74 in., and the thermometer at 76°, required the true altitude of the moon's centre.

The object in this example being the moon, it is necessary to compute her semidiameter and parallax in altitude at the instant of observation since these elements, for the moon changes sensibly in a very short time. The semidiameter of the moon at noon and midnight is given in the Nautical Almanac for every day in the year, at page III. of each month, and the difference between these will be the variation of the semidiameter in 12 hours. Therefore we must say as 12^h : the variation in 12^h : : the interval between the preceding noon or midnight and the instant of observation : the variation of the semidiameter in that interval; the fourth term of this proportion added to or subtracted from the semidiameter at the preceding noon or midnight, according as the semidiameter is observed from the numbers in the almanac to be increasing or decreasing, will give the semidiameter at the instant of observation.

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[*] The limb of the sun or moon is the edge or border of the disc.

[†] Take out the refractions for 16° 30′ of altitude from the table, then the diff. for 1′ of altitude in the column adjoining, multiplying the latter by 2, and subtracting the product from the refraction for 16° 30′; the result will be that for 16° 32′, when the barometer is at 30 in. and the thermometer at 50°. The correction for refraction is always subtractive.

[‡] The barometer standing at 29 in. the number taken from the column entitled cor. for + 1 must be subtracted from the refraction or added to the altitude, the atmosphere being less dense than in its medium state.

[§] The thermometer standing at 8° above its medium state, the atmosphere is more rare, and the number taken from the column Diff. for 1° Fah., after being multiplied by 8, must be subtracted from the refraction, or added to the altitude.

[‖] Table XXXIV., the parallax in alt. for 10° is 9″, and for 20° is 8″. Therefore for 16° by proportion it is 8″·4. This correction for par. in alt. is always additive.

In a similar manner must the moon's horizontal parallax, which is given for every noon and midnight on the same page of the Nautical Almanac, be reduced by proportion to the time of observation.

The computation of these elements is as follows:

Mean time of observation at the station	11^h	44^m	55^s
Add longitude of the place of observation	4	56	4
Corresponding mean time at Greenwich	16	40	59
Time after midnight Gr. June 23d	4	40	59

Semidiameter previous midnight, June 23d (Naut. Alm.)	14′	50″	·3
Semidiam. noon (June 24th),	14	48	·1
Variat. in 12^h		2	·2
∴ 12^h : 2″·2 : : $4^h\ 40^m\ 59^s$[*] :		0	·8
Semidiam. at midnight (23d),	14	50	·3
Semidiam. at time of obs.,	14	49	·5
Augmentation for 29° of alt.[†]		+ 7	·8
Apparent semidiam. to obs.,	14	56	·3
Subtract contraction,[‡]		— 1	·1
True semidiam. to observer,	14	55	·2

Horizontal parallax preceding midnight, June 23d (Naut. Alm.)	54′	27″	·3
Hor. par. noon (24th),	54	19	·1
Var. in 12^h		8	·2
∴ 12^h : 8″·2 : : $4^h\ 40^m\ 59^s$:		3	·1
Hor. par. at midnight,	54	27	·3
Hor. par. at time of obs.,	54	24	·2
Ditto in seconds,		3264″	·2
Diminut. of par. for lat. 41°[§]		— 4	·7
Hor. par. at station,		3259″	·5

[*] This is the interval from midnight at Greenwich to the instant of observation.

[†] Table XXXIII. This augmentation is in consequence of the moon being nearer to the observer, as it approaches the zenith. See p. 272.

[‡] This is occasioned by the effect of refraction, which is to make every vertical arc, such as the vertical semidiameter of the sun or moon, appear shorter in the heavens than it really is. This will obviously be the case, because the lower extremity of the arc is more elevated by refraction than the higher, and consequently the two extremities are brought nearer together, and thus the arc is shortened. The contraction is obtained from Tab. XXXII.

[§] Table XXXV., see p. 275, last paragraph of Art. 109.

Observed double altitude ☽'s U. L.	58° 14′ 00
Index error subtractive,	15″
Double altitude corrected for index error,	58 13 45
Half this is the obs'd altitude of ☽'s U.L.	29 6 52·5
Corrected semidiameter,	14 55·2
Apparent altitude ☽'s centre,	28 51 57·3

FOR THE PARALLAX IN ALTITUDE.[*]

App. alt. ☽'s centre,	28° 51′ 57″·3	cos 9·94239
Horizontal parallax at station,	3259″·5	log. 3·51315
Parallax in altitude,	2854″·6	3·45554

App. alt. ☽'s centre,	28° 51′ 57″·3
Refraction,	— 1 44 ·9
Barometer,	— 2 ·7
Thermometer,	+ 5 ·5
Parallax in altitude,	+ 47 34 ·6
True alt. of cent. from cent. of the earth,	29 37 49 ·8

These two examples will serve for specimens of the corrections to be applied to an observed altitude, in order to deduce from it the true altitude of the body's centre. In the case of the moon, the corrections, when the utmost accuracy is sought, are rather numerous, as the last example shows.

But in finding the latitude at sea, it is usual to dispense with some of these, more especially with the corrections for temperature, for the contraction of the moon's semidiameter, and for the spheroidal figure of the earth; because an error of a few seconds in the true altitude will introduce no error worth noticing in the resulting latitude. When, however, the object of the observer is to deduce the longitude of the ship, all the data, furnished by observation, should be as accurate as possible; for the problem is one of such delicacy that by neglecting to allow for the influence of temperature would alone introduce in some cases an error of from 30 to 40 miles in the longitude.

When the object observed is a star, several of the foregoing corrections vanish; the only corrections in this case requisite are those for dip and refraction, modified as usual for the temperature.

111. *To determine the latitude at sea from the meridian altitude of any celestial object whose declination is known.*

The determination of the latitude, by a meridian altitude, is the most easy and safe method of finding that element; the observations and subsequent calculations being few, are readily performed, and with but little liability to error in the result; this method, therefore, is always to be preferred at sea, unless clouds obscure the meridian whilst other portions of the heavens are left visible.

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[*] See p. 275.

The declination of the object observed is supposed to be given in the Nautical Almanac, when it culminates or makes its meridian transit at Greenwich; its declination when it culminates at the meridian of a ship, may be found by means of the longitude by account,* which will always be sufficiently accurate for this purpose, although it should differ very considerably from the true longitude, because declination changes so slowly that even an error of an hour in the longitude would cause an error in the declination too small to deserve notice.

The declination being the distance of the object from the equator, and the observed altitude, properly corrected, being the distance of the same object from the ship's zenith, the distance of the zenith from the equator, that is, the latitude, immediately becomes known.

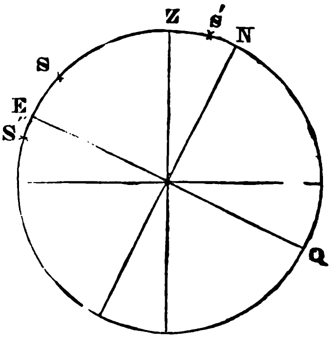
Let the full circle in the diagram be the meridian.

1. Let s be the object observed, the zenith z being to the north of it, and the object itself north of the equator, EQ, then the latitude EZ is equal to the zenith distance, or co-altitude ZS + the declination ES, and it is north.

2. Let s′ be the object, still north of the equator, but so posited that the zenith is south of it, then the latitude EZ is equal to the difference between the zenith distance s′z, and declination s′E, and is still north.

3. Let now the object be at s″, south of the equator, and the zenith to the north of the object, then the latitude EZ is equal to the difference between the zenith distance s″z and declination s″E, and it is north.

We have here assumed the north to be the elevated pole, but if the south be the elevated pole, then we must write south for north, and north for south. Hence the following rule for all cases.



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* For this purpose the variation of the declination in 1 hour, which is given in the Nautical Almanac for the sun, must be multiplied by the longitude in hours and fractions of an hour, and the product added or subtracted will produce the declination at the time of meridian transit at the ship.

Call the zenith distance north or south, according as the zenith is north or south of the object.

If the zenith distance and declination be of the same name, that is, both north or both south, their sum will be the latitude; but if of different names, their difference will be the latitude, of the same name as the greater.

EXAMPLE.

1. Ship Admiral, from New York to Havre, at sea Jan. 4th, 1850. Longitude 25° W.

Observed merid. alt. ⊙'s lower limb,	18°	54′	20″
Dip (for height of 17 feet)		— 3	57
App. alt. ⊙'s L. L.	18	50	23
Refraction,		— 2	49
Parallax in alt.			+ 8
Semidiameter,		16	17·3
⊙'s true alt.	19	3	59·3
⊙'s zenith dist. N.	70	56	00·7
⊙'s declination[*] S.	22	43	37·9
Latitude, N.	48	12	22·8

2. At sea Jan. 11th, 1850. Long. 2° W.

Observed alt. ⊙'s L.L.	18°	4′	00″
Allowing for semidiameter (Dip 3′ 26″) parallax, &c.	+	12	29

Required the latitude.

Ans. 49° 57′ 29″ N.

3. On the 1st of January, 1850, the meridian altitude of Capella was 27° 35′, the zenith being south of the star, and the height of the eye 22 feet; required the latitude.

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[*] This is obtained by taking out from the Nautical Almanac, the declination for apparent noon, p. I., which is 22° 44′ 5″·2. Then computing the change in declination for 1⅔ hours, the time in which the sun is passing from the meridian of Greenwich to that of the ship in long. 25° W., by multiplying the number 16″·38, found in the column in the Almanac entitled Diff. for 1 hour by 1⅔. The product 27″·3, subtracted from 22° 44′ 5″·2, because the declination is decreasing will give the declination at the meridian transit of the sun at the ship.

Observed altitude, ...	27° 35′ 0″
Dip, ...	— 4 30
Apparent altitude, ...	27 30 30
Refraction, ...	— 1 51
True altitude, ...	27 28 39
Zenith distance, ...	62 31 21 S.
Star's dec. (Nautical Almanac), ...	45 50 20 N.
Latitude, ...	16 41 1 S.

4. Suppose that the altitude of the moon, as given in Example 2, p. 277, was observed when the moon was upon the meridian, required the latitude of the place of observation.

The true altitude of the moon's centre being known, after applying the corrections as at p. 278, it remains to find her declination at the instant of observation. The Nautical Almanac gives the moon's declination for every even hour of the day of every day, on pages V. to XII. of each month, and the variation in declination for 10^m of time. The required declination would therefore be computed as follows:

☽'s dec. June 23d, at 16^h (Nautical Almanac),	19° 36′ 48″·8
Diff. dec. for 41^m,*	1 35
Dec. at the inst. of observation, ...	19 38 23 ·8 S.
☽'s zenith dist. = (90° — 29° 37′ 49″·8†) .	60 22 10 ·2 N.
Latitude required, ...	40 43 46 ·4

If the time of observation were not known, it could be computed from the fact that the moon is on the meridian.

The moon passed the Meridian of Greenwich

June 23d (Nautical Almanac, p. IV.), at . .	11^h 33^m 0^s
June 24th, " "	12 21 7
The interval between the two transits is . .	24 48 7

That is in 24^h 48^m 7^s the moon is retarded in coming to the meridian, by her proper motion from W. to E. 48^m 7^s

$$\therefore 24^h\ 48^m\ 7^s : 48^m\ 7^s :: 4^h\ 56^m\ 4^s‡ : 9^m : 39^s$$

---

* The time of obs. was 40^m 59^s past 6^h or nearly 41^m. The Naut. Alm. gives 23″·17 diff. of dec. for 10^m ∴ 10^m : 23″·17 : : 41^m : 1′ 34″ the change in dec. in 41^m, which as dec. is increasing must be added. Second differences are not used.

† This is the true alt. of the ☽'s centre from the centre of the earth, p. 279.

‡ This is the longitude of the place of observation.

This last number is the retardation of the moon in passing from the meridian of Greenwich to that of the place of observation. The moon having crossed the meridian at Greenwich at 11^h 33^m on the 23d, will cross that of the station 9^m 39^s later, so that the time of meridian transit at the station will be 11^h 36^m 39^s.*

It saves trouble to note the time of meridian transit by a watch, or still better by a chronometer, keeping Greenwich time.

SCHOLIUM.

These examples will, no doubt, be found sufficient to put the student in possession of the method of applying the various corrections to the observed meridian altitude of a celestial object, in order to deduce from it the latitude of the ship. But it should be remarked, that in most works on Nautical Astronomy, subsidiary tables are inserted for the purpose of abridging some of the foregoing corrective operations; such tables, therefore, offer very acceptable aid to the practical Navigator. Bowditch's Navigator is the most complete work of the kind.

It should also be observed here, that in the preceding examples the celestial object is supposed to be on the meridian *above* the pole; that is, to be higher than the elevated pole. But, if a meridian altitude be taken below the pole, which may be done if the object is *circumpolar*, or so near to the elevated pole as to perform its apparent daily revolution about it without passing below the horizon, then the latitude of the place will be equal to the sum of the true altitude, and the codeclination or polar distance of the object; for this sum will obviously measure the elevation of the pole above the horizon, which is equal to the latitude.†

112. *To determine the latitude at sea, by means of two altitudes of the sun, and the time between the observations.*

In the preceding article we have shown how to determine the latitude of the ship by the meridian altitude of the sun, or of any other heavenly body, whose declination may be found. But, as already remarked, the object we wish to observe may be obscured when it comes to the meridian, and this may happen for many days together, although it may be frequently visible at other times of the day. As therefore the opportunity for a meridian observation cannot be depended upon, it becomes an important problem to determine the latitude at sea, by observations made out of the meridian; and considerable attention has accordingly been paid, by scientific persons, to the method of finding the latitude by "double altitudes," and various tables have been computed to facilitate the operation. But the direct method, by spherical trigonometry, though rather long, involving three spherical triangles, will be more readily remembered, and more easily applied by persons familiar with the rules and formulas of trigonometry than any indirect or approximate process; we shall therefore explain the direct method.

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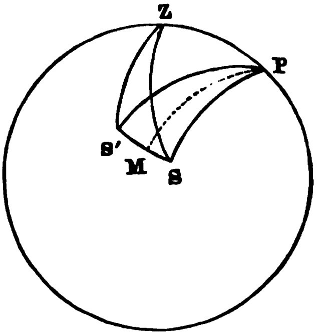
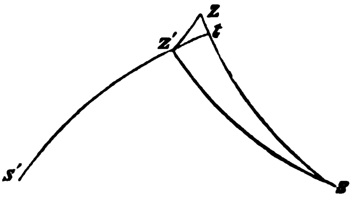
* This differs slightly from the time of observation given. The moon changes so rapidly in declination that her greatest altitude is not always the meridian altitude.

† That the elevation of the pole above the horizon is equal to the latitude of the place is evident from the fact that the zenith is 90° from the horizon, and the pole 90° from the equator.



Let P be the elevated pole, Z the zenith of the ship, and S, S′ the two places of the sun, when the altitudes are taken. Then, drawing the great circle arcs as in the figure, we shall have these given quantities, viz., the co-declinations PS, PS′; the coaltitudes ZS, ZS′, and the hour angle SPS′, which measures the interval between the observations; and the quantity sought is the colatitude ZP. Now, in the triangle PSS′, we have given two sides and the included angle to find the third side SS′, and one of the remaining angles, say the angle PSS′. In the triangle ZSS′ we have given the three sides to find the angle S′SZ; having then the angles PSS′, S′SZ, the angle ZSP equal to their difference, becomes known, so that we have, lastly, two sides and the included angle in the triangle ZSP, to find the third side ZP.

Before the application of the trigonometrical process, the observed altitudes must, of course, be reduced to the true altitudes, as in the preceding examples. Moreover, as the ship most probably sails during the interval of the observation, an additional reduction becomes necessary, as follows: Let Z be the zenith of the ship, and S the place of the sun, at the first observation Z′ and S′ the same at the second. Then the angle Z′ZS will represent the bearing of the ship's path from the sun, which may be observed with the compass; considering



this angle as a course, and the distance sailed, zz′, as the corresponding distance, find by the table (or by the formula zz′ cos z′*zt*) *zt* which subtracted from zs will give z′s nearly, which, instead of zs, should be used with z′s′ in the solution before given. This must be subtracted from the first zen. dist. if the angle z′*zt* is less than 90°; but it must be added when the angle exceeds 90°. If the angle is 90°, no correction for the ship's change of place will be necessary.

Where great accuracy is aimed at, account should be taken of the ship's change of longitude during the interval of the observations; when converted into time it must be added to the interval of time between the observations when the ship has sailed eastward, and subtracted when she has sailed westward. This correction is very easily applied.

Having thus mentioned the necessary preparative corrections, we shall now give an example of the trigonometrical operation.

EXAMPLE.

Let the two zenith distances corrected be (see last figure but one), zs = 73° 54′ 13″, zs′ = 47° 45′ 51″, the corresponding declinations 8° 18′ and 8° 15′ north, and the interval of time three hours; to determine the latitude.

Considering ss′ to be the base of an isosceles spherical triangle, of which one of the equal sides is ½ (PS + PS′)* = 81° 43′ 30″, and the vertical angle equal to 3^h or 45°, let the perpendicular PM be drawn, then we have in the triangle PMS right angled at M, PS = 81° 43′ 30″, and $P = \frac{45^\circ}{2} = 22^\circ\ 30'$, given, to find SM = ½ ss′ as follows.

1. TO FIND SS′ FROM THE TRIANGLE PMS.

sin PS	81°	43′	30″	9·99545
sin P	22	30	0	9·58284
sin SM	22	15	11·4	9·57829
			2	
ss′ =	44	30	22·8	

---

* Which we may, without sensible error, where the base is so small.

TO FIND PSS′ FROM THE TRIANGLE PSS′.

sin SS′	44° 30′ 22·8″	arith. comp.	0·15429
sin PS′	81 45 0	. . .	9·99548
sin SPS′	45 0 0	. . .	9·84948
sin PSS′	86 38 53	. . .	9·99925

This angle is acute like its opposite side (see p. 196).

III. TO FIND ZSS′ IN THE TRIANGLE ZSS′.

ZS′	47° 45′ 51″		
sin ZS	73 54 13	. arith. comp.	0·01737
sin SS′	44 30 22·8	. arith. comp.	0·15429
	2)166 10 26·8		
½ sum =	83 5 13·4		
sin (½ sum — ZS)	9 11 0·4	. . . .	. 9·20302
sin (½ sum — SS′)	38 34 50·6	. . . .	. 9·79492
			2)19·16960
sin ½ ZSS′	22 36 26·4	. . . .	. 9·58480

∴ ZSS′ = 45° 12′ 52·8″
PSS′ = 86 38 53
PSZ = 41 26 0·2

IV. TO FIND THE TWO UNKNOWN ANGLES OF THE TRIANGLE ZSP.

cos ½ (ZS + PS)	77° 48′ 52″	ar. comp.	0·67555	ar. comp. sin	0·00990
[*] cos ½ (ZS ∼ PS)	7° 49′ 17″	. . .	9·99594	sin	9·13381
cot ½ PSZ	20° 43′ 3″	. . .	10·42228	. . . .	10·42228
tan ½ (ZPS + PZS)	85° 23′ 35″	. .	11·09377	tan ½ (P ∼ Z) 20° 12′ 32″	9·56599
	20° 12′ 32″				

SPZ = 105° 36′ 7″
ZPS = 65° 11′ 3″

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[*] This sign ∼ is employed to express the difference between two quantities, whichever may be the greater.

V. TO FIND ZP IN THE TRIANGLE ZSP.

$\cos \frac{1}{2}(Z - P)$	20° 12′ 32′ ar. comp.	0·027593
$\cos \frac{1}{2}(Z + P)$	85° 23′ 31″	8·904822
$\tan \frac{1}{2}(ZS + PS)$	77° 48′ 52″	10·665658
$\tan \frac{1}{2} ZP$	21° 37′ 14″	9·598073
$ZP =$	43° 14′ 28″	

Upon the same principles may the latitude be determined from the altitudes of two fixed stars, taken at the same time; in this case S, S′, in the preceding figure, will represent the two stars: PS, PS′, their known polar distances, and the angle SPS′, the difference of their right ascensions; the same quantities are therefore given as in the case of the sun, but, as in the case of two stars, PS, PS′, may differ very considerably, SS′ cannot be considered as the base of an isosceles triangle, but must be computed from the other two sides and their included angle.

For other modes of determining the latitude, see the next Appendix.

ON FINDING THE LONGITUDE.

The determination of the longitude of a place always requires the solution of these two problems, viz.: 1st, to determine the time at the place at any instant; and, 2d, to determine the time at the first meridian, or that from which the longitude is estimated, at the same instant; for the difference of the times converted into degrees, at the rate of 15° to an hour, will obviously give the longitude.

When the latitude of the place is known (and it may be found by the methods already explained), the time may be computed from the altitude of any celestial object whose declination is known; for the coaltitude, codeclination, and colatitude, will be three sides of a spherical triangle given to find the hour angle, comprised between the codeclination and the colatitude.

(See Art. 84.) The following example will illustrate the mode of proceeding.

At Columbia College, January 13th, 1850, the double altitude of the sun's lower limb was observed by reflexion from Mercury to be 42° 29′. Thermometer 40°, and Barometer 30 in.

Time by the watch, $10^h\ 6^m\ 10^s$ A.M.

Index error of the sextant, 52″ additive.

Latitude of station 40° 42′ 40″.

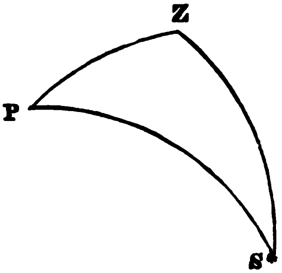
Longitude from Greenwich in time, $4^h\ 56^m\ 4^s$.

Required correct time of Observation and error of the watch.

Equation of time at ap. noon, January 13th, 1850, 9′ 00″ •27.
Difference per hour =0 •926.*
Sun's declination at ap. noon, January 13th, 1850, 21° 29′ 19″ •5 S.
Difference per hour = 26″•1.

Observed double altitude, ...	42°	29′		
Index error of sextant. Additive (see 2d note, p. 290).			52″	
Double altitude corrected for index error, . . .	42	29	52	
Altitude, ...	21	14	56	
Refraction (Th. 40), (B. 30), Table XXX., . . .		—2	35	•5
Sun's Parallax in Altitude, Table XXXIV., . .			8	
Semidiameter (Nautical Almanac)		16	16	•8
⊙'s true altitude.corrected for refraction, parallax, and semidiameter	21	28	45	•3
⊙'s zenith distance (90° — Altitude)	68	31	14	•7

Approx. time at station, ...	10^h	6^m	10^s	A.M.
Longitude from Greenwich in time, .	4	56	4	
Time at Greenwich, ...	3	2	14	
Equation of time, subtractive, . .		9	3 •08	
Time after apparent noon at Greenwich,	2	53	10 •92	
⊙'s declination at time of observation†,	21	28	4 •18	S.



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* This must be multiplied by the time after apparent noon at Greenwich, found below, reduced to hours and decimals of an hour, and the product added to the equation of time at noon above, to obtain the equation of time at the instant of observation.

† This is computed from the data in the third and fourth lines from the top of the page, in the same manner as the equation of time.

PS or z = ⊙'s N.P.D = (90° + D) =	111° 28′ 4″·18	ar. co. log. sin 0·03122
PZ or s = Colatitude	49 17 20	ar. co. log. sin 0·12033
ZS or p = Zenith distance	68 31 14 ·7	
S	229 16 38 ·88	
½ S	114 38 19 ·44	
½ S — z	3 10 15 ·26	log. sin 8·74285
½ S — s	65 20 59 ·44	log. sin 9·95850
		2)18·85290
½ P 15° 29′ 1″		log. sin 9·42645

Hour angle = 30° 58″ 2′.

	^{h}	^{m}	^{s}
$\frac{P}{15}$ = Hour angle in time, or apparent time before noon,	2^h	3^m	52^s ·1
Subtract hour angle from	12		
Apparent time, A.M. by observation, . . .	9	56	7 ·9
Equation of time, additive, ...		9	3 ·08
Mean time by observation, ...	10	5	10 ·98
Mean time by watch, ...	10	6	10
Error of watch (too fast)			59 ·02

2. Ship Admiral at sea, March 5th, 1850, at 10 o'clock, A. M.

Observed altitude sun's lower limb, ...	27° 44
Height of eye above the level sea, 16 feet.	
Time at Greenwich, by mean of three chronometers .	12^h 41^m 12^s
Lat. at time of obs. north, ...	49° 54′ 00″

EXTRACTS FROM NAUTICAL ALMANAC.

At mean noon.

March 6th, ⊙'s semidiameter,	16′ 7″·9;	eq. of time,	— 11^m 30^s·98*
7th, " "	6 7 ·6	" "	— 11 16 ·56

⊙'s dec. at apparent noon Gr. (6th) . . . 5° 40′ 48″·3 S.

Diff. for 1^h, 58″·20†

Required the time of observation.

Ans. ^{h} ^{m} ^{s}

3. March 12th, at 4 P.M. Astronomical Account.

Alt. of the sun's lower limb, ...	18° 42′ 40″
Time at Greenwich by mean of chronometers, . .	7^h 51^m 30^s
Lat. of ship at time of obs.	41° 41′ 30″

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* To be subtracted from mean time.

† The dec. is of course diminishing till the equinox March 21st.

At mean noon.

March 12th, eq. of time — 9^{m} $59^{s}{\cdot}14$ semidiam. 16′ 6″·3*
" 13th, " " — 9 42·71 " 16 6
☉'s dec. at app. noon, March 12th (Gr.) 3° 20 11· 7 S.
Var. of dec. in 1^{h} is 59″·02.

Required the time. *Ans.* ^{h} ^{m} ^{s}.

To find the time at Greenwich requires the aid of additional data, besides those furnished by observations made at the place. The Greenwich time may, indeed, be obtained at once, independently of any observations at the place, by means of a chronometer, carefully regulated to Greenwich time, provided it be subject to no irregularities after having been once properly adjusted. A ship furnished with such a timepiece always carries the Greenwich time with her, and the longitude then becomes reduced to the problem of finding the time at the place.

EXAMPLE.

Time computed by an altitude of the sun, as at p. 288, was	9^{h} 45^{m} 10^{s}
Chronometer showed Greenwich time at the instant of observation to be	0 50 20
Difference of longitude of the place of observation from Gr. in time, ...	3 5 10
To convert time to space multiply by	15
Longitude of the ship west of Greenwich, ...	46° 17′ 30″

The same method applies to examples 2 and 3, p. 289.

Still, however, as the most perfect contrivance of human art is subject to accident, and the more delicate the machine the more liable is it to disarrangement, from causes which we may not be able to control, it becomes highly desirable, in so important a matter as finding the place of a ship at sea, to be possessed of methods altogether beyond the influence of terrestrial vicissitudes, and such methods the celestial motions alone can supply.

The angular motion of the moon in her orbit is more rapid than that of any other celestial body, and sufficiently great to render the portion of her path passed over in so short a time as two or three seconds, a measurable quantity even with a small portable instrument (the sextant).†

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* In computing the ☽'s semi-diameter and parallax, second differences need not be used, the consequent error being less than 0″·1.

† THE SEXTANT

is constructed upon the same principles as the quadrant. It consists of a graduated

It is obvious, therefore, that if the distance of the moon's centre from any celestial body, in or near her path, be computed for any Greenwich time, and this distance be found the same as that given by actual observation at any place, then the difference between the time of observing

brazen arc of 60°, numbered double, however, for the same reason as in the quadrant, or to 120°, called the limb, upon which moves a vernier attached to one end of an index, the other end of which is at the centre of the arc. Upon the latter, in a direction parallel with it, and perpendicular to the plane of the limb, is a mirror called the index glass, adjustable by three screws to perpendicularity with the plane of the limb. Opposite the index glass, and parallel with its plane when the index is at zero, is another glass, half mirror and half transparent, called the horizon glass. A small telescope parallel to the plane of the limb is placed before the horizon glass, and directed so as to look through the latter.

There are three adjustments. 1. *To make the index glass perpendicular to the plane of the limb.* This is done by moving forward the index to the middle of the limb, then looking with the naked eye into the index glass; if the part of the limb seen by reflection appear in the same plane with the part seen direct, the index glass is perpendicular to the plane of the limb; if not, it must be adjusted.

2. *To make the horizon glass perpendicular to the plane of the limb.*—The index glass having been adjusted, hold the instrument in a vertical position, and bring the direct and reflected images of the same object to coincide; if this can be done exactly, no adjustment is required, but if one image appear at the right or left of the other, the horizon glass must be adjusted by a screw or screws attached to it for the purpose.

3. *To make the axis of the telescope parallel to the plane of the limb.*—Bring the images of two objects which are more than 90° apart, to coincide upon one of the parallel wires in the telescope, and then by turning the instrument in the hand a little, make the objects appear on the other wire. If the coincidence remains, the position of the telescope is correct; if not, it must be adjusted by the screws of the ring into which the telescope is screwed. N. B. There are usually two telescopes accompanying the sextant, the one an inverting or astronomical telescope, and the other not. There is also a plane tube without glasses, and either of the three may be screwed into the same ring.

There are darkening glasses to be used in observing the sun, four near the index glass, and three before the object glass. They are red and green, of different shades. The latter color is particularly good to take off the glare of the moon. The paler one before the horizon glass may sometimes be used with advantage to take off the glare of the horizon below the sun, occasioned by the reflection of that luminary from the small rippling waves. N. B. The parallelism of the surfaces of the darkening glasses should be tested by inverting them, and observing if the coincidence of objects be preserved.

When the index stands at zero the direct and reflected images of the same object ought to coincide. If not, there is *index error*, the amount of which is determined by observing how far the index stands from zero when the direct and reflected images coincide. The best mode of determining the index error is by measuring the diamethe phenomenon and the time at Greenwich, when it was predicted to happen, will give the longitude of the place of observation. Now, in the Nautical Almanac the distances of the moon from the sun, and from several of the fixed stars near her path, are given for every three hours of apparent Greenwich time, and for several years to come, and the Greenwich time, corresponding to any intermediate distance, is obtainable by simple proportion; so that by means of the Nautical Almanac we may always determine the time at Greenwich when any distance observed at sea was taken.* (See Nautical Almanac, pp. XIII. to XVIII. inclusive.)

The distances inserted in the Nautical Almanac are the true angular distances between the centres of the bodies, the observer being considered as at the centre of the earth, and to the true distance therefore every observed distance must be reduced; it is this reduction which constitutes the trigonometrical difficulties of this problem. And it consists in *clearing the lunar distance from the effects of parallax and refraction*; how to do this, it is now our business to explain.

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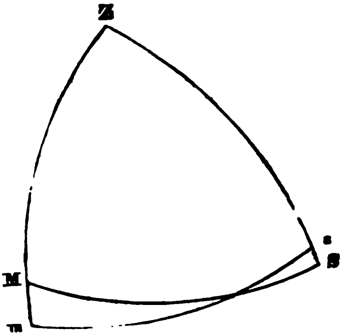
ter of the sun by moving the index both forward and backward, the limb being graduated a short distance behind the zero for the purpose. Half this difference of the two measures will be the index error.

To measure an angle with the sextant, bring the two objects, the line joining which subtends the angle, the one as seen direct, and the other by reflection, to coincide, by holding the instrument so that the plane of the limb passes through them, and moving the index forward; the reading shown by the index will be the angle required.

* The proportion would run thus: As the difference between two lunar distances given in the Almanac, the one greater, the other less than that observed with the sextant, is to 3 hours, so is the difference between the distance observed with the sextant corrected for refraction, &c., and one of those in the Alm., to the difference between the time of observation and the time given by the Almanac corresponding to the latter lunar distance. The Nautical Almanac gives the proportional logarithm of the quotient of the first term of the above proportion, divided by the second term which is constant, viz., three hours. If the distance between the moon and a star increased or decreased uniformly, the Greenwich time corresponding to a given distance would be strictly correct; but an inspection of the columns of proportional logarithms in the ephemeris, will show that this is not the case. A correction for second differences, or for the irregularity in the lunar distances, must therefore be applied.

At page 602 of the Nautical Almanac for 1850 is given, besides the ordinary rule and an example under it, a full explanation of the method of employing a table contained in the Almanac for computing the correction on account of second differences in finding the Greenwich time. The theory of second differences has been given (Algebra, Art. 235). At the scientific meeting in New Haven, August, 1850. Prof. Chauvenet of the U. S. Naval Academy presented improvements in the formulæ and tables for lunars, which it is to be hoped will be perfected and published.

Let m, s, be the observed places of the moon and sun, or of the moon and a fixed star, and let M, S, be their true places. M will be above m, because the moon is depressed by parallax more than it is elevated by refraction; but S will be below s, because the sun is more elevated by refraction than it is depressed by parallax. Observation gives the apparent distance ms, and the apparent zenith distances zm, zs: by applying the proper corrections to these latter, we also deduce the true zenith distances ZM, ZS, and with these data we are to determine the true distance, MS, by computation.



Put d for the apparent distance.*
D true distance.
a, a' apparent altitudes.*
A, A′ true altitudes.

Then in the triangle MZS we have (Art. 82), R being 1,

$$\cos \text{Z} = \frac{\cos \text{D} - \sin \text{A} \sin \text{A}'}{\cos \text{A} \cos \text{A}'};\dagger$$

and in the triangle mzs

$$\cos \text{Z} = \frac{\cos d - \sin a \sin a'}{\cos a \cos a'};$$

hence, for the determination of D, we have this equation, viz.,

$$\frac{\cos \text{D} - \sin \text{A} \sin \text{A}'}{\cos \text{A} \cos \text{A}'} = \frac{\cos d - \sin a \sin a'}{\cos a \cos a'};$$

from which we immediately get

$$\cos \text{D} = (\cos d - \sin a \sin a') \frac{\cos \text{A} \cos \text{A}'}{\cos a \cos a'} + \sin \text{A} \sin \text{A}'$$

---

[*] In observing d with the sextant, it is the nearest point of the limb of the moon, which is made to coincide with the other heavenly body, and in observing a with the quadrant, it is the limb also which is made to coincide with the horizon; so that d and a must be corrected for the semidiameter of the moon; similar remarks apply to the sun, if he be the other heavenly body.

[†] Observe that A and A′ are the complements of ZM and ZS.

But $\cos a \cos a' - \sin a \sin a' = \cos (a + a')$ Art. 70; transposing $\cos a \cos a'$, and substituting the value of $- \sin a \sin a'$ thus obtained, we have

$$\cos \mathrm{D} = \frac{\cos d + \cos (a + a') - \cos a \cos a'}{\cos a \cos a'} \cos \mathrm{A} \cos \mathrm{A}' + \sin \mathrm{A} \sin \mathrm{A}'$$

Dividing the last term of the numerator by the denominator, the quotient is -1; then observing that $-\cos \mathrm{A} \cos \mathrm{A}' + \sin \mathrm{A} \sin \mathrm{A}' = -\cos (\mathrm{A} + \mathrm{A}')$ and that $\cos d + \cos (a + a') = 2 \cos \frac{1}{2} (a + a' + d) \cos \frac{1}{2} (a + a' \sim d)$ Art. 83, we have

$$\cos \mathrm{D} = \frac{2 \cos \frac{1}{2} (a + a' + d) \cos \frac{1}{2} (\overline{a + a' \sim d}) \cos \mathrm{A} \cos \mathrm{A}'}{\cos a \cos a'} - \cos (\mathrm{A} + \mathrm{A}') \quad (1)$$

EXAMPLE.

1. Suppose the apparent distance between the centres of the sun and moon to be 83° 57′ 33″, the apparent altitude of the moon's centre 27° 34′ 5″, the apparent altitude of the sun's centre 48° 27′ 32″, the true altitude of the moon's centre 28° 20′ 48″, and the true altitude of the sun's centre 48° 26′ 49″; then we have

$$d = 83° \, 57' \, 33'', \; a = 27° \, 34' \, 5'', \; a' = 48° \, 27' \, 32''$$
$$\mathrm{A} = 28° \, 20' \, 48'', \; \mathrm{A}' = 48° \, 26'' \, 49';$$

and the computation for D, by formula (1), is as follows:

	d	83°	57′	33″		
	a	27	34	5	ar. comp. cos	052339
	a'	48	27	32	ar. comp. cos	178383
		2)159	59	10	log. 2	301030
½ sum		79	59	35	cos	9·239969
½ sum $\sim d$		3	57	58	cos	9·998959
	A	28	20	48	cos	9·944527
	A′	48	26	49	cos	9·821719
(Reject 40 from index)						$\bar{1}$·536926 = log. ·344292 +
A + A′		76	47	37	nat. cos	·228460
True distance 83° 20′ 54″					nat. cos	·115832

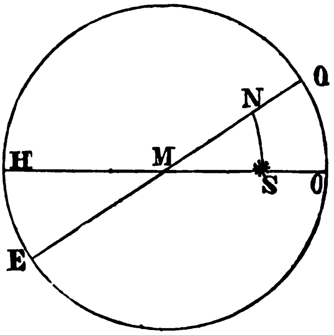
By glancing at the formula (1), we see that 30 must be rejected from the sum of the above column of logarithms, to wit, 20 for the two ar.

comp^{s}. and 10 for R, which must be introduced into the denominator, in order to render the expression homogeneous, so that the logarithmic line resulting from the process is 9·536926. Now, as in the table of log. sines, log. cosines, &c., the radius is supposed to be 10^{10}, of which the log. is 10, and in the table of natural sines, cosines, &c., the rad. is 1, of which the log. is 0; it follows that when we wish to find, by help of a table of the logarithms of numbers, the natural trigonometrical line corresponding to any logarithmic one, we must diminish this latter by 10, and enter the table with the remainder. Hence the sum of the foregoing column of logarithms must be diminished by 40, and the remainder will be truly the logarithm of the natural number represented by the first term in the second number of the equation (1). If this natural number be less than nat. cos $(A + A')$, which is to be subtracted from it, the remainder will be negative, in which case D will be obtuse.

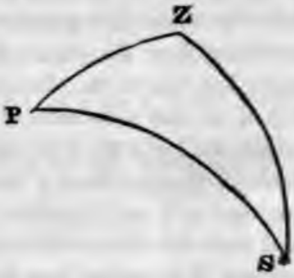
VARIATION OF THE COMPASS.

114. We shall conclude this part of our subject by briefly considering the methods of finding the variation of the compass, or the quantity by which the north point, as shown by the compass, varies easterly or westerly from the north point of the horizon.

The solution of this problem merely requires that we find by computation, or by some means independent of the compass, the *bearing* of a celestial object, that we observe the bearing by the compass, and then take the difference of the two. The problem resolves itself, therefore, into two cases, the object whose bearing is sought being either in the horizon or above it: in the one case we have to compute its *amplitude*, and in the other its *azimuth*.



The computation of the amplitude is simply determining the hypothenuse of a right angled triangle MSN, of which one side is given, viz. the declination NS of the object, as also the angle opposite to it, viz. the colatitude M. The computation of the azimuth requires the solution of an oblique spherical triangle, the three sides being given to find an angle; the three given sides are the colatitude PZ, the zenith distance of the object ZS, and its polar ance PS, and the azimuth being measured by the angle at the zenith z, posite the polar distance, this is the gle sought. We shall give an example in each of these cases of the lem.



EXAMPLES.

1. In January, 1830, at latitude 27° ′ N., the rising amplitude of Alde-aran was by the compass* E. 23° 30′ N.; required the variation.

From the Nautical Almanac, it appears that the declination of Aldebaran at the given time was 16° 9′ 37″ N., therefore since, by Napier's rule, Rad. × sin dec. = sin. amp. × cos lat., the computation is as follows:

sin declination	16°	9′	37″	9·44455
cos. latitude	27	36		9·94753
sin amplitude E.	18	18	17 N.	9·49702
Magnetic amplitude E.	23	30	0 N.	
Variation	5	11	43	

As the object is farther from the magnetic east than from the true east, the magnetic east has therefore advanced towards the south, and therefore the magnetic north towards the east; hence the variation is 5° 11′ 43″ E.

2. In latitude 48° 50′ north, the true altitude of the sun's centre was 22° 2′, the declination at the time was 10° 12′ S., and its magnetic bearing 161° 32′ East. Required the variation.

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* The compass amplitude must be taken when the apparent altitude of the object is equal to the depression of the horizon.

⊙'s polar distance	100°	12′			
sin zenith distance	67	58		ar. comp.	0·03294
sin colatitude	41	10		ar. comp.	0·18161
	2)209	20			
½ s.	104	40			
sin (½ s. — zen. dist.)	36	42		. . .	9·77643
sin (½ s. — colat.)	63	30		. . .	9·95179
					2)19·94277
sin ½ azim.	69°	25′	40″	. . .	9·97138
			2		
⊙'s true azimuth N.	138	51	20	E.	
observed azimuth N.	161	32	0	E.	
Variation	22	40	40	West.	

The variation is west, because the sun's observed distance from the north, measured easterly, being greater than its true distance, intimates that the north point of the compass has approached towards the west.

3. In latitude 48° 20′ north, the star Rigel was observed to set 9° 50′ to the northward of the west point of the compass; required the variation, the declination of Rigel being 8° 25′ S.

Variation, 22° 33′ West.

APPENDIX TO PART V.

A very portable and accurate instrument for the measurement of altitudes of the heavenly bodies is

THE CIRCLE OF REFLECTION.*

This instrument is constructed on the same principles as the sextant, the only difference being that the circle, as its name imports, has a limb which is a complete circumference, measuring on the doubling principle of optics already explained 720° instead of 360°.

By having two verniers 180° apart, the instrument corrects its own eccentricity, and by having three, as in Troughton's construction, for error of figure and division, to some extent also. By reversing the face of the instrument the angle may be taken on what is called the off arc, that is, on the other side of the zero. This, by taking the mean, corrects for index error, so that with three verniers six readings may be obtained for one angle.

Dolland's circle has an inner movable circle, with a vernier upon it, in consequence of which it admits of the repeating process explained for the theodolite, at p. 242.

The horizon glass and telescope are attached firmly to the inner movable circle, which has a clamp screw and screw of slow motion. The index glass is attached to an arm which moves freely around the centre, and is unconnected with the inner circle, telescope and horizon glass. This arm has a vernier and screw for clamping it to the outer circle, and a screw of slow motion.

The repeating process is conducted as follows: Place the zero of the vernier of the inner circle clamped accurately at 720° on the outer, and move the free index carrying the other vernier forward until the two objects are brought in contact, as in observing with the sextant. Leaving this index fast to the limb, unclamp the inner circle, which carries the telescope and horizon glass, and move it forward also, not merely by the same amount, which would bring back the horizon glass to parallelism again with the index glass, but move it twice the angular distance necessary for this purpose ; the horizon glass will now be inclined to the index glass, just as much as when the free index was first moved forward to produce the contact of the objects, but the inclination will be the other way. The contact may again be produced between the objects by the tangent screw of the inner circle, the telescope being ented to the other object if the face of the circle be continued one way, but to the same object if the face of the circle be reversed. The reading now of the vernier attached to the inner circle would be twice the angle between the objects. If now the free index be moved forward a distance equal to the angle between the objects, the horizon and index glasses will be parallel again; and if it be moved forward still further by the same amount, the glasses will be inclined to each other, exactly as at irst, and the contact of the objects may be made with the screw of slow motion ttached to the free index, the face of the circle being as at first. Again, the inner circle is to be moved forward as before, over twice the angle between the objects, the face of the circle being inverted, the telescope directed always to one object, and the contact made with the screw of slow motion attached to the inner circle. The process above described is to be repeated until the vernier of the inner circle approaches near, or is a little past the 720 point again. The reading at which it stands, divided by twice the number of times that the inner circle has been moved forward, will give the angle subtended by the two objects corrected for all errors of division, centering and observation. If the angle between the objects be changing, as is often the case with celestial objects, the times of each contact of the object should be noted. The n of the times, divided by the number of contacts, will be the mean time corres-nding to the mean angle obtained, as already described. The Sextant and Reflecting Circle are used for taking altitudes on land, by the aid of a basin of mercury, called an artificial horizon. The telescope is presented to the reflected image of the sun or other heavenly body, seen in the mercury, and the angle between this and the sun in the heavens measured by moving the index forward. This angle will be double the altitude of the sun.

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* Angles in any plane may be observed with the Circle of Reflection as with the Sextant.

The following example of an observation of the altitude of the sun for time will illustrate the mode of using this instrument.

Observation with Repeating and Reflecting Circle, and Box Chronometer, of the sun's lower limb.

	$2^h\ 29^m\ 33^s$	
	" 30 00·5	
	" 30 19	
	" 30 43·5	
	" 31 06.5	
Times of contacts.	" 31 34·5	117° 36′ 10″ Vernier reading.
	" 31 59·5	720°
	" 32 15	10)837 36 10
	" 32 36	
	" 32 55·5	2)83 45 37
	10)313 03	41 52 48·5
	$2^h\ 31^m\ 18^s{\cdot}3$	

The mean of the angles being 41° 52′ 48″·5, and of the times, $2^h\ 31^m\ 18^s{\cdot}3$, the former is the apparent altitude at the instant expressed by the latter, with no error of excentricity or graduation, and a probable error of observation $\frac{1}{10}$ that which would have been obtained without the repeating process.

VARIOUS MODES OF DETERMINING LATITUDE.

LATITUDE BY A SINGLE ALTITUDE.

The data for this problem are the declination, the altitude, and the hour angle. We have thus three elements of the triangle SPZ (p. 296), viz., SP, SZ, and the angle P, known, to calculate a fourth. This method requires an accurate knowledge of the time.

If the latitude be known nearly, which it generally is, by means of the dead reckoning, it may be accurately found as follows. From (7) of Art. 72 we have

$$\cos \mathrm{A} = 1 - 2 \sin^2 \tfrac{1}{2} \mathrm{A} \qquad (1)$$

which substituted in the formula deduced at Art. 82, written thus

$$\cos a = \cos b \cos c + \sin b \sin c \cos \mathrm{A} \qquad (2)$$

gives by (8) of Art. 70,

$$\cos a = \cos (b - c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2} \mathrm{A} \qquad (3)$$

substituting in (3) the sides and angles of ZPS, calling the polar dist. π, the zenith dist. ζ, and the colatitude λ, that equation becomes

$$\cos \zeta = \cos (\pi - \lambda) - 2 \sin \pi \sin \lambda \sin^2 \tfrac{1}{2} \mathrm{P} \qquad (4)$$

But if ζ' denote the meridian zenith distance of the heavenly body

$$\zeta' = \pm (\pi - \lambda) \qquad (5)$$

Hence (4) becomes

$$\cos \zeta = \cos \zeta' - 2 \sin \pi \sin \lambda \sin^2 \tfrac{1}{2} \mathrm{P}$$

or, by transposition,

$$\cos \zeta' = \cos \zeta + 2 \sin \pi \sin \lambda \sin^2 \tfrac{1}{2} \mathrm{P} \qquad (6)$$

the formula for use in which

ζ' is meridian zenith dist.
ζ is observed zenith dist.
π is polar distance of the object observed.
λ is the approximate colatitude.
P is the hour angle.

The use of formula (6) is facilitated by Table XXIII., commencing p. 154 of Bowditch's Navigator, entitled log. rising, in which log. $2 \sin^2 \tfrac{1}{2} \mathrm{P}$ is calculated for every value of the hour angle P. To this it is only necessary to add the logarithms of the sines of the polar distance and approximate colatitude, and we have the difference between the observed and meridian altitudes, or the correction to be applied to the former to obtain the latter, from which the latitude is calculated as at p. 280.

A formula may be derived from either of the forms (B) of Art. 86. Applied after clearing of fractions to the present triangle it becomes

$$\sin^2 \tfrac{1}{2} \mathrm{P} \sin \pi \sin \lambda = \sin \tfrac{1}{2} (\zeta + \lambda - \pi) \sin \tfrac{1}{2} (\zeta + \pi - \lambda)$$

or from (5) above

$$\sin^2 \tfrac{1}{2} \mathrm{P} \sin \pi \sin \lambda = \sin \tfrac{1}{2} (\zeta + \zeta') \sin \tfrac{1}{2} (\zeta - \zeta')$$

$$\sin \tfrac{1}{2} (\zeta - \zeta') = \sin^2 \tfrac{1}{2} \mathrm{P} \sin \pi \sin \lambda \ \operatorname{cosec} \tfrac{1}{2} (\zeta + \zeta') \qquad (7)$$

letermine $\zeta-\zeta'$, the correction for the zenith distance, if in the .d memb alue of ζ' be used, which would be given by the approximate value of the latitude.

CIRCUM-MERIDIAN ALTITUDES.

If the heavenly body be near the meridian, P and $\zeta-\zeta'$ will be very small, and mall arcs in place of their sines, to which they are sensibly equal (7) nsidering $\zeta=\zeta'$ in the second member)

$$\tfrac{1}{2}(\zeta-\zeta')=\tfrac{1}{4}P^2\sin\pi\sin\lambda\operatorname{cosec}\zeta' \qquad (8)$$

in which the value of the first member is expressed in terms of the radius as unity. express it in seconds of arc, its value, as well as that of P must be divided by sine of $1''$, which may be regarded as the length of $1''$, in terms of radius as nity. (8) thus becomes, striking out at the same time the common factor $\frac{1}{2\sin 1''}$

$$\zeta-\zeta'=\frac{P^2}{2\sin 1''}\sin\pi\sin\lambda\operatorname{cosec}\zeta' \qquad (9)$$

which $\zeta-\zeta'$ is expressed in seconds of arc, as is also P the hour angle. If p enote the hour angle in time, $15p$ must be substituted for P, and (9) becomes

$$\zeta-\zeta'=\frac{225p^2}{2\sin 1''}\sin\pi\sin\lambda\operatorname{cosec}\zeta' \qquad (10)$$

The value of $\zeta-\zeta'$ for the same station and star being proportional to p^2, if it were calculated for a value of $p=1^m$, it might be found for any other value of p, by multiplying this by p^2 in minutes of time. Table XXXII. of Bowditch's Navigator contains values of $\zeta-\zeta'$ for $p=1^m$ for all latitudes and all declinations less than 24°, so that entering this table with the declination of the star and proximate latitude of the station as arguments, the number taken from the table multiplied by the square of the hour angle, in minutes and decimals of time, which is given in Tab. XXXIII. of Bowditch, for every value of p up to 13^m, will produce the correction to be applied to the observed altitude, to obtain the meridian altitude.

$\zeta-\zeta'$ may be calculated more accurately by means of a table (Tab. XXXVI.), adapted to formula (9) above, in which $\frac{P^2}{2}$ is equal to the versed sine of P, as may be seen by referring to (2) on p. 94, from which, using only two terms on account of the smallness of P, we have

$$\cos P=1-\frac{P^2}{2}\ \therefore\ \operatorname{vers}P=1-\cos P=\frac{P^2}{2}$$

Table XXXVI. gives vers P and the arithmetical complement of the logarithm of the sin $1''$ is 9·3144.

N. B. The correction $\zeta-\zeta'$ is subtractive from ζ the observed zen. dist. to obtain ζ' the merid. zen. dist. or it is additive to the observed altitude to obtain meridian altitude.

If several altitudes or zenith distances were observed near the meridian, and each reduced to the meridian by the above formula, the mean of the latitudes thence derived would be the true latitude more nearly in proportion to the number of observations.

But the mean of the values for $\zeta - \zeta'$, obtained from (10), since for each the quantities entering into (10) are the same except p^2, may be obtained by taking the mean of the values of p^2, and multiplying this by the constant factors. And if the mean of the values of $\zeta - \zeta'$ be subtracted from the mean of the values of ζ, the same latitude will result as if the latitude were calculated for each observed zenith distance, and the mean of all the latitudes taken, but with much less computation.

EXAMPLE.

Observed circum-meridian altitudes of ☉'s lower limb, at head of Upper Mistigougiche Lake, latitude about 48°, July 24th, 1841, with Repeating and Reflecting Circle, Artificial Horizon, and Box Chronometer.

SUN'S CENTRE ON THE MERIDIAN BY CHRONOMETER AT $11^h\ 12^m\ 48^s{\cdot}1$[*], BAROMETER 28·9, THERMOMETER 69.

Times of obs.	Hour Angles, p.	p^2 Tab. XXXIII.	Tab. XXXVI.
11^h 09 55·5	$2^m\ 52^s{\cdot}6$	$8^m{\cdot}3$	78
10 31	2 17 ·1	5 ·2	50
10 47	2 1 ·1	4 ·1	39
11 09	1 39 ·1	2 ·7	26
11 30	1 18 ·1	1 ·7	16
11 51·5	0 56 ·6	0 ·9	8
12 13·5	0 34 ·6	0 ·3	3
12 36·5	0 11 ·6	0 ·0	0
12 53	0 4 ·9	0 ·0	0
13 11·5	0 23 ·4	0 ·1	1
13 24	0 35 ·9	0 ·4	3
13 44	0 55 ·9	0 ·9	8
		12)24 ·6	12)232
Vernier Reading 19° 26′ 20″	mean	2 ·05	19

$\zeta - \zeta'$ for $p = 1^m$ in lat. 48°, dec. 19°, Table XXXII. $2''{\cdot}6$

$\zeta - \zeta'$ for mean of hour angles = product $5''{\cdot}3$

The times in the first column above are the instants of contact of the images as observed with the reflecting circle, according to the method described at p. 300.

The numbers in the 2d column are obtained by subtracting those of the first from $11^h\ 12^m\ 48^s{\cdot}1$, the time at which the sun is on the meridian. The numbers in the 3d column are the squares of those in the 2d, expressed in minutes and tenths. The mean of the values of p^2 is $2^m{\cdot}05$, which, multiplied by $2''{\cdot}6$, the value of $\zeta - \zeta'$ lat. 48° and dec. 19°, gives $5''{\cdot}3$, the correction required, by which the mean of the observed altitudes is to be increased, to produce the meridian altitude.

The fourth column is for another mode of computation, the numbers in it being from XXVI.* The computation of formula (9) by the aid of this n is as rs:

---

[*] This of course depends on the equation of time, and the error of the chronometer.

Mean of vers.	24	log.	1·2787
Latitude,	48° 6′	log. cos	9·8243
Declination,	19 10	log. cos	9·9752
Zenith Dist.,	29 12	log. cosec	0·3116
Constant 1″	ar. co	log. sin	9·3144
Correction ζ′—ζ	5″·06	log.	0·7042

The reading of the vernier at the end of the observations 19° 26′ 20″ was after two complete revolutions of the circle; therefore the whole angle passed over is d by adding this to 1440° = twice 720°, and the double altitude is obtained ding this by twice the number of times that the inner circle has been moved rward, or by the whole number of contacts of images, in this case 12.

	°	′	″
	19°	26′	20″
	1440		
	12)1459	26	20
Double altitude,	121	37	11·7
Altitude,	60	48	35·85
Reduction to Meridian,			5·06
Semidiameter,		15	46·4
	61	4	27·31
Refraction—Parallax,		—	25·40
	61	4	1·91
Declination,	19	10	47·3
Colatitude,	41	53	14·61
Latitude,	48	6	45·39

The following method of determining the latitude by means of three altitudes taken near the meridian, has been given by Mr. Littrow, of Vienna.

Let A be the first observed altitude corrected for dip, refraction, parallax, and semidiameter, and T the time.

$A + a$ the second observed altitude, corrected also, and

$T + t$ the corresponding time.

$A + a'$ the third altitude, and $T + t'$ the time.

Let A' be the meridian altitude, and $T + T'$ the corresponding time. Then the variations of the altitude near the meridian being as the squares of the variations of the hour angle nearly, we have

$$A' - A : A' - a : : T'^2 : (T' - t)^2$$

$$\therefore \quad aT'^2 = (A' - A)(2T't - t^2) \qquad (1)$$

$$\text{Similarly } a'T'^2 = (A' - A)(2T't' - t'^2)$$

$$\text{By division } \frac{a}{a'} = \frac{2\,T't - t^2}{2\,T't' - t'^2} \text{ and } \therefore \ T' = \tfrac{1}{2}\,\frac{at'^2 - a't^2}{at' - a't} \qquad (2)$$

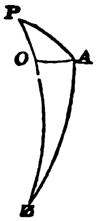
$$\text{And from (1) } A' - A = \frac{aT'^2}{2\,Tt' - t^2} \qquad (3)$$

---

* For a table which gives the value of the whole fraction in (9) and (10), see Lee's Tables and Formulæ, p. 64, Part III.

From (2) may be obtained the value of T′, which added to T gives T + T′, the time when the object observed was on the meridian, and from (3) may be obtained the value of A′ — A, and by adding A, the value of A′ the meridian altitude becomes known, from which the latitude is to be computed as at Art. 111.

Determination of Latitude by an Altitude of the Pole Star out of the Meridian.



Let P be the pole, Z the zenith, and consequently PZ the meridian; A the place of the pole star at the time that its altitude a is observed. Denoting its polar distance PA by π, drawing the arc AO perpendicular to PZ, expressing PO by y, and the hour angle by p, in the right angled triangle PAO, which may be regarded as plane from being so small, we have

$$\text{AO} = \pi \sin p$$

In the right angled triangle ZAO, ZA and ZO are nearly equal; denoting their difference by x, we have $\text{ZO} = 90 - a - x$, and by Napier's rules,

$$\cos \text{ZA} = \cos \text{ZO} \cos \text{AO}$$

or,

$$\sin a = \sin (x + a) \cos (\pi \sin p) \qquad (1)$$

But

$$\sin (x + a) = \sin x \cos a + \cos x \sin a = \sin a + x \cos a \qquad (2)$$

since $\cos x = 1$, and $\sin x = x$ because x is so small. Also [p. 94 (2)]

$$\cos (\pi \sin p) = 1 - \tfrac{1}{2} \pi^2 \sin^2 p \qquad (3)$$

Uniting (2) and (3), (1) becomes

$$\sin a = (\sin a + x \cos a)(1 - \tfrac{1}{2} \pi^2 \sin^2 p)$$

Performing the multiplication indicated in the second member, and omitting the term $\tfrac{1}{2} x\pi^2 \cos a \sin p$ on account of the smallness of the quantities x, π, and p, we deduce

$$x = \tfrac{1}{2} \pi^2 \tan a \sin^2 p$$

But in the right angled triangle PAO

$$\tan y = \tan \pi \cos p$$

And (p. 94, note),

$$\tan \pi = \pi + \tfrac{1}{3} \pi^3$$

Also, (p. 95, note),

$$y = \tan y - \tfrac{1}{3} \tan^3 y$$

$\therefore y = (\pi + \tfrac{1}{3} \pi^3) \cos p - \tfrac{1}{3} \pi^3 \cos^3 p$, or $y = \pi \cos p + \tfrac{1}{3} \pi^3 \sin^2 p \cos p$* omitting the powers of π higher than π^3.

---

* Since $\sin^2 p = 1 - \cos^2 p$.

Denoting the ude required by λ, it is evident (see second note, p. 283), that

$$\lambda = a + x - y$$

$$\therefore \lambda = a - \pi \cos p + \tfrac{1}{2} \pi^2 \tan a \sin^2 p - \tfrac{1}{3} (\pi \cos p) (\pi \sin p)^2 \qquad (4)$$

To have λ in seconds of arc we must substitute for λ, $\lambda \sin 1''$, and for π, $\pi \sin 1''$. Thus (4) becomes, after dividing by $\sin 1''$

$$\lambda = a - \pi \cos p + \tfrac{1}{2} \sin 1'' \pi^2 \tan a \sin^2 p - \tfrac{1}{3} \sin^2 1'' (\pi \cos p) (\pi \sin p)^2 \qquad (5)$$

$$\log. \tfrac{1}{2} \sin 1'' = \bar{6}{\cdot}3845449$$

$$\log. \tfrac{1}{3} \sin^2 1'' = \overline{12}{\cdot}8940285$$

Formula (5) is that upon which the rule p. 619 of the Naut. Alm. of 1850 is founded. The argument for the first correction $-\pi \cos p$, in the formula, of the observed altitude, necessary to be applied to obtain the altitude of the pole equal to the latitude of the place, is the siderial time, upon which p or the hour angle evidently depends, the declination being regarded as constant. There will be two arguments of the second correction $\frac{1}{2} \pi^2 \tan a \sin^2 p - \frac{1}{3} (\pi \cos p) (\pi \sin p)^2$ to wit, the siderial time and the altitude. The third correction is the change in the value of the t, $-\pi \cos p$, which alone is of sufficient magnitude to be sensibly affected by the change in the right ascension and declination of the pole star. Denoting the changes by da and $d\delta$, the change in $-\pi \cos p$ occasioned by the latter will evidently be

$$d\delta \cos p$$

and that occasioned by the former da will be

$$-\pi (\cos (p - da) - \cos p) = -\pi \sin da \sin p = -\pi da \sin 1' \sin p$$

as will be seen by developing $\cos (p - da)$, and making $\cos da = 1$ on account of the smallness of da. Or the above expression may be written

$$5380 \times 0{\cdot}000073\, da \sin p = 0{\cdot}4\, da \sin p$$

The third correction is sometimes positive, sometimes negative, but always less than 1′, and therefore

1′ + the third correction,

which is the quantity given in the table in the Naut. Alm., is always positive, and hence the necessity of subtracting 1′ according to the rule given in that work.*

The instrument employed for measuring altitudes in a fixed observatory is

THE MURAL CIRCLE,

so called because it is sustained against a wall about four feet thick, and wide and high enough to allow ample space for the circle, which is usually five or six feet in diameter. The graduation is not where it usually is, but upon the outer rim, the surface of which is normal to the plane of the circle. The degrees run from zero to 360. Exterior to the circle, and fastened against the wall, are six microscopes, pointing towards the centre of the circle for reading the graduated rim. They are placed at equal distances, viz. 60° apart. The circle turns round past these microscopes, which are stationary, upon a very strong axis, fastened solidly to one side of the circle at the centre, where the axis is about six inches thick. This axis passes entirely through the stone wall, tapering to about three inches or less at the other side of the wall. Against this small end of the axis, screws work both laterally and vertically, by means of which the circle is adjusted to the plane of the meridian. This adjustment is effected by the aid of what is called a ghost apparatus. Before describing this it will be necessary to mention that the telescope of the instrument is attached firmly to the outer face of the circle, in the direction of the diameter, to which it is equal in length, and turns with the circle on its axis. The ghost apparatus consists of two small microscopes, placed one at each end of the telescope, on the outside of its tube, and pointing horizontally and parallel to the plane of the circle, when the telescope is vertical. From a bracket at the top of the wall, moving outward and inward by a screw, is suspended a plumb line, which passes down directly before the object glasses of the microscopes, between them and a small mother of pearl disc, on which is a black spot, called a ghost, which is brought exactly on the line of vision of the microscopes, so as to be covered by the plumb line. This being done, if now the plumb line be moved out by means of the screw in the bracket, so as to clear everything, and the telescope be reversed by turning the circle 180° on its axis, then, if the plumb line, when brought back to cover one of the ghosts covers also the black spot in the other, the axis of the circle is horizontal. If not, half the correction must be made by the adjusting screws of the axis at the rear of the wall, and the other half by moving one of the ghosts. This latter movement is effected by turning on its axis the tube containing the mother of pearl disc, which tube is in the prolongation of the tube of the microscope, the connexion between them being broken, to permit the plumb line to pass in. The black spot of the ghost is at one side of the centre of the mother of pearl disc, so that as the tube containing the disc turns, the black spot is carried from side to side, through a small space, sufficient to bring it into coincidence with the plumb line, the latter remaining stationary, and covering the black spot of the other ghost.

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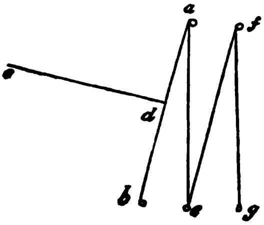
* The above method of finding the latitude is so convenient as to be worthy of particular attention. It is not confined to any precise hour, as is the case with meridian altitudes, but may be employed during the whole night, whenever the pole star is visible. An officer of the British Navy has, by means of it and a quadrant with a spirit level attached, the horizon being invisible, been able to run his frigate boldly up the English Channel, at the rate of eleven knots an hour, in a dark squally night, which allowed only occasional glimpses of the pole star between flying clouds.

Captain Porter, U.S.N., is in the habit of running his steamer along the Florida Reefs by the same means.

The process above described must be repeated till the plumb line coincides with the black spot of the ghosts in both positions of the telescope.



The rationale of this adjustment may be made evident by the following diagram.

Suppose the plane of the diagram to be a vertical plane through the centre of the mural circle, and perpendicular to its plane; *ab* its section with the plane of the circle, *cd* perpendicular to *ab*, the axis on which the circle turns, *ae* the first position of the plumb line coinig with the spots on the ghosts *a* and *e*. When the telescope is reversed by urning the circle on its axis *cd*, the ghost *e* will take the position *f*, and the ghost *a* will come to *b*, and the plumb line suspended from *f* will be in the direction *fg*. The lower ghost *b* will thus be the whole distance *bg* away from the plumb line *fg*. Half the correction necessary to bring it into coincidence with the plumb line, must erefore be made by lowering the end *c* of the axis, which is firmly fixed to the cle *ab*, till the point *b* comes to *e*,* which will render the axis *cd* horizontal, and other half by moving *b*, now at *e*, to *g*, or the ghost *f*, together with the plumb line, to *a*. The axis *cd* is thus made horizontal, and the circle *ab* vertical t being difficult in practice to lower the point *c* just enough to make half the cortion which would render *cd* horizontal, the process has to be repeated. Each l will bring the axis nearer to the horizontal position required, so long as only part of the correction necessary to bring the ghosts into alignment with the plumb is made by the motion of the point *c* of the axis *cd*, by means of the screws ch work against this point.

The axis being thus rendered horizontal, the instrument may be collimated by the method described at p. 148, and may be adjusted to the meridian, or its deviation the meridian determined by the method of high and low stars, given at p. 157.

measure the polar distance of a star with this instrument, the reading must be aken as indicated by an index or pointer, fastened to the wall a little outside he circle, when the telescope points to the pole of the heavens, and afterwards taken when the telescope points to the star, the circle having been moved on its axis past the index, which remains stationary. The difference between these two readings will be the polar distance of the star.

The first of the above mentioned readings is called the polar zero of the instrument. It may be found by taking the readings with the telescope pointing to a circumpolar star, at both its superior and inferior transits. The middle point between these readings or half their sum will be the polar zero.

In taking the observation the horizontal wire in the axis of the telescope is made to thread the star with great exactitude. This wire does not require collimating, but the vertical wires do, for the reason stated at p. 160.

Each degree of the graduated limb is divided into halves and quarters, or 15′; these again into 3 parts, or 5′ spaces. The reading of the degree is taken by the index or pointer above mentioned, the minutes and seconds with one of the microscopes, and the seconds with all six microscopes, the mean of the seconds given by the whole six being the number of seconds adopted. The construction and mode of using these microscopes must now be described. The wires of one of these microscopes, which are of the finest spider's web, cross each other at a very acute angle, with a horizontal line, and are placed in a frame in the tube of the microscope at the focus of its object glass. This frame is movable up and down, by means of a screw placed under the tube of the microscope, and working into it. A single turn of the screw moves the point of intersection of the wires over $\frac{1}{5}$ of one of the 5′ spaces, and the screw head is made large, and its circumference divided into 60 parts, so that the number of turns and fractions of a turn of the screw necessary to bring the wires to one of the dividing lines of the 5′ spaces will show how many minutes and seconds the intersection of the wires stood past this division. To save the trouble of counting the number of turns of the screw, a notched scale is placed vertically a little on one side, at the focus of the microscope, so as to be magnified by its eye-glass. A pin, which points horizontally past the scale, towards the intersection of the wires of the microscope, moves over one notch at each turn of the screw. The zero of the notched scale at which the pin ought to stand at the commencement of the observation, is marked by a hole at the middle of the scale.

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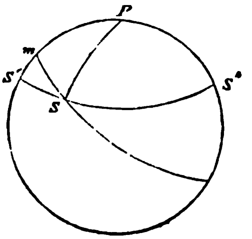
* Or rather till *e* and *b* come together, the motion being about the point *d*, *a* moving half the distance *be* to the left, and *b* half the same distance to the right.

To recapitulate, the reading of the degrees, minutes, and seconds will be taken as follows: the degrees by the index or pointer, then looking into the microscope, the degree nearest which the pointer stands will be recognised by a round indentation in the metal, near a long dividing line; the half degree is marked by a dividing line of the same length, without any indentation; the quarter degree by a shorter line, and the 5′ spaces, of which there are three in each quarter degree, by shorter lines still. If the intersection of the wires stand exactly upon one of these dividing lines, the reading will be so many degrees, halves, quarters, and five minutes; but if the intersection of the wires of the microscope be a little past one of the dividing lines on the limb, the intersection must be brought back to the dividing line by turning the screw which moves the frame containing the wires. The number of turns of the screw necessary for this purpose, indicated by the number of notches passed over on the scale by the pin within, will be so many additional minutes to be added to the reading as it now stands, and the seconds will be obtained from the screw head by noting at what numbers upon it its index stands.

The mural being used in the same observatory ordinarily with a transit instrument, by observing the time of transit of the star over the wires of the mural, and comparing the result with the transit observation corrected as explained under the head of that instrument, the hour angle of the star when it crosses the imaginary middle wire will be known; or it can be computed from a high and low star observed with the mural itself, as explained for the transit instrument. From this hour angle the error in the observed altitude of the star arising from the deviation of the instrument from the meridian, may be obtained as follows:

REDUCTION TO MERIDIAN OF AN OBSERVATION MADE WITH THE MURAL CIRCLE.

Let the full circle be the meridian of the station, s′ss″ the diurnal path described by the star under observation, *ms* the arc of a great circle of which the horizontal wire is a portion, s the place of the star when observed. Then in the triangle MSP, right angled at *m*, (since the meridian is a vertical circle)



$$\cos \mathrm{P} = \tan \mathrm{P}m \cdot \cot. \mathrm{PS} = \frac{\sin \mathrm{P}m}{\cos \mathrm{P}m} \cdot \frac{\cos \mathrm{PS}}{\sin \mathrm{PS}}$$

$$1 - \cos P = 1 - \frac{\sin Pm \cdot \cos PS}{\cos Pm \cdot \sin PS}$$

$$= \frac{\cos Pm \sin PS - \sin Pm \cos PS}{\cos Pm \sin PS} = \frac{\sin (PS - Pm)}{\cos Pm \sin PS}.$$[*]

But $1 - \cos P = 2 \sin^2 (\frac{1}{2} P)$, see (8) Art. 72.

Since $PS = PS'$, $PS = Pm$ very nearly, whence $PS - Pm = s'm$, and the hour angle PS' is generally very small.

$$\therefore \quad 2 \sin^2 (\tfrac{1}{2} P) = 2 (\sin 1'')^2 \tfrac{1}{4}\dagger\ P^2 = \frac{s'm \cdot \sin 1''}{\frac{1}{2} \sin 2\, PS\ddagger}$$

rhence

$$s'm = \text{Reduction} = \tfrac{1}{4} \sin 2\, PS,\ P^2 \sin 1''$$

ich is expressed in arc. To express it in time make $P = 15\ p$, whence

$$\text{Reduction} = \frac{225}{4} \sin 2\, PS\ p^2 \sin 1'' = 6{\cdot}4356974 \sin 2\, PS\ p^2$$

hich p is the hour angle in seconds of time, and PS the polar distance of the The correction or reduction does not become of any sensible magnitude, till hour angle exceeds 5 seconds. The correction or reduction is always to added to the apparent polar distance, if the star has a N.P. distance less than 90°, ut subtracted if greater than 90°. This may be seen from the diagram, and still better on a globe; the great circle of which the horizontal wire is a portion, intersecting the meridian N. of the diurnal circle of the star, when its N.P.D. is less than 90°, and S. when it is greater.

Instead of the polar zero, a horizontal zero has been more commonly used. It is obtained by taking the reading with the horizontal wire bisecting the star, as seen direct on one night, and as seen by reflexion from mercury on the following night; half the difference of the two readings will be the meridian altitude of the star, from which and the latitude the polar dist. or declination may be obtained as explained at p. 150. Within a few years past a third zero, which may be called the nadir zero, has been employed. This is obtained by means of a collimating eye-piece and basin of mercury, placed under the object glass of the telescope, turned vertically downward, as described at p. 161, note. When the direct image of the horizontal wire coincides with that seen by reflexion, the reading will be the nadir zero, and the difference between this and the reading when the telescope points to a star crossing the meridian, will be the distance of the star from the nadir, which is the supplement of its zenith distance, from which the polar dist. or dec. may be obtained, as already explained.*

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[*] See (7) Art. 70.

[†] The sine of a very small arc differs insensibly in length from the arc itself, and its value may be found by multiplying the sin of 1'', taken from the tables, by the number of seconds in the arc. This serves to express the length of the arc in the same terms as the radius, sine, cosine, &c., are expressed, so that these quantities can enter the same formula.

[‡] See form. (1) Art. 71.

TRIAL METHOD OF DETERMINING BOTH LATITUDE AND TIME.

Supposing neither the chronometer error nor the latitude to be exactly known, but the altitudes of two stars A and B, differing 90° in azimuth be observed, and assume successively two trial latitudes, computing the chronometer error by each latitude, and by the altitude of each of the stars. Suppose the first assumed latitude for example to be 45° 15′, and the resulting chronometer error by star A to be $46^{s}{\cdot}2$, and by star B, $50^{s}{\cdot}2$; the discordance is 4^{s}. Suppose the second assumed latitude to be 45° 25′, and the resulting chronometer error by A to be $53^{s}{\cdot}4$, by B $37^{s}{\cdot}4$; the discord now is 16^{s} the opposite way. Then 10′ of latitude has changed the discord 20^{s}; therefore a change of 2′ of latitude will remove the first discord of 4^{s}, and the latitude is therefore 45° 17′; and since 10′ altered the error by A $7^{s}{\cdot}2$, and by B $12^{s}{\cdot}8$, 2′ will alter the former by $1^{s}{\cdot}44$, and the latter by $2^{s}{\cdot}56$, and both agree in giving the error $47^{s}{\cdot}64$.

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* The circle and transit instrument are combined in one, in the German observatories. The instrument which results is called a *transit circle* or *meridian circle*. Both observations, to wit, that for the right ascension and that for the declination of the object, are taken simultaneously.

PART VI.

GEODESY.

PART VI.

GEODESY.

Geodesy is a higher kind of surveying, which takes into account the curvature of the earth's surface. It has for its object to determine, with the utmost possible accuracy, the geographical positions of points on the earth's surface by the process of triangulation already repeatedly described in this work, but requiring for the present purpose certain modifications, which it will be the object of the following pages to unfold.

The triangles composing the chain best fulfil their destination when the largest possible, and nearly equilateral. The sides are ordinarily from ten to fifty miles, and limited only by the want of distinct vision with the instruments, or interruptions from the nature of the ground. The primary chain being finished, a *secondary* chain of smaller triangles, having their vertices within the larger, is surveyed, and a still smaller chain of *tertiary* triangles. A great number of small bases and points of reference are thus determined, from which the surveys with the plane table or compass may originate, to complete the map in all its details. One of the chief problems, after the triangulation is finished, is the determination of the difference of latitude and longitude between the vertices of the triangles, so that when the absolute latitude and longitude of some of the vertices shall have been determined by astronomical observations, that of all the others may be known by *differentiation*. Previous to the explanation of the method employed for this purpose, it will be necessary to give some account of the mode of conducting the triangulation and, first, of the

MEASUREMENT OF BASES.

The measured base, from which the triangulation commences, should be selected upon ground which will admit of its extending several miles. Its length should be ascertained with the greatest care, and for this purpose it has been customary to use metallic rods, allowing for their expansion and contraction from changes of temperature, the amount of which been previously determined by experiment, with a thermometer which was always observed when making the measurements.* For greater accuracy in placing the ends of the rods together, an optical contact has been employed. This is produced by placing horizontally upon tressels the rods, one a little above, and its end projecting over that of the next, a notch being cut out of the ends of the rods, across which a thread is horizontally stretched; a microscope supported on a stand, from which an arm holding the microscope projects over the ends of the rods, so that the microscope can be placed vertically over the threads, looking down upon them; the rods are then moved by screws in the tressels, till the threads are seen to coincide in the microscope, or till each coincides th a mark on the stand below. The optical contact is then complete.

The best *base apparatus* now in existence, probably, is that employed upon the coast survey of the United States, and of this we shall attempt a more particular description. It is made self-adjusting under changes temperature. The measuring rods consist each of two bars, the one ve the other; the lower of brass and the upper of iron, which is less ansible. The two bars are connected firmly by a cross-piece at one nd, but allowed each to expand freely, being unconnected at the other, at which, however, there is a short lever of the second order placed vertically, having its fulcrum or extremity at the end of the lower bar, on which it works as a hinge, and its point of resistance at the end of the upper bar, the lever continuing above this, and having the power applied horizontally to its upper end, through a steel rod, which projects horizontally from the next measuring rod, above which it is sustained by a short support. The contact between this steel rod and the upper end of the lever is that of a blunt knife edge against a plane of agate. The other end of the steel rod works freely against the lower end of a very short vertical lever of the first order, the upper end of which supports a spirit level. The end of the spirit level farthest from the steel rod has a tendency to fall, occasioned by a counterpoise weight, which projects from it, but this is prevented by pressure of the steel rod against the lower end of the short lever, on the top of which the spirit level rests. This delicate mode of contact, first suggested by Bessel, depends on the sensibility of the spirit level.† The tressels on which the rods rest, at two points only, are strongly trussed, and their upper parts at least metallic, and admit, by means of screws, of the various motions required in placing the apparatus. In measuring the base line, the measuring rods need not be placed exactly in a horizontal position, but the measure they give can easily be reduced to a horizontal one, if only the inclination of the rod to the horizon be known*.

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* For each degree of the centigrade thermometer, platina expands 0·000008565 of its dimensions in every direction, iron 0·000010666, and brass 0·000017843. For 1° Fah. the expansion is for brass, 0·00001050903, iron 0·000006963535.

† The effect of this arrangement is evident; the whole rod, composed of the two bars, lengthens by the effect of heat, but the lower more than the upper bar, so that

The contrivance to indicate this consists of a sector, the plane of which is vertical, attached to the rods and graduated; a brass radius of this sector, moving about its centre, and supporting a spirit level, stands at the zero of the sector when the rod and spirit level are both horizontal. When the rod is inclined, the number of degrees passed over on the sector, in making the spirit level horizontal, shows the inclination of the rod.

---

the upper end of the vertical lever, which makes the contact, is thrown back. The ratio of the length of the arms of the lever will depend on the ratio of expansibility of brass and iron.

* The formula would evidently

$$\text{B} = l \cos a$$

B being the horizontal measure required, l the oblique measure given by the apparatus, and a its inclination shown by the sector. But a being very small, it is better to compute the correction $l - \text{B}$ to be subtracted from l to obtain B. It is

$$l\,(1 - \cos a) = 2\,l \sin^2 \tfrac{1}{2}\,a = \tfrac{1}{2}\,l\,a^2 \sin 1'.$$

or,

$$l - \text{B} = 0{\cdot}00000004231\ a^2\,l$$

or,

$$\log.\,(l - \text{B}) = 2{\cdot}626422 + 2\ \log.\ a + \log.\ l.$$

a being very small. In practice it is customary to tabulate this formula.

The base must next be reduced to the level of the neighboring sea. Let ρ be the radius of the earth (or better the normal) for the level of the sea; $\rho + h$ the radius for the level of the base, h being the mean height of the ground on which the base is measured above the level of the sea, and b the reduced base.

Then, since similar arcs are as their radii, we have

$$\frac{\text{B}}{b} = \frac{\rho + h}{\rho}\ \therefore\ \text{B} - b = \frac{\text{B}h}{\rho + h} = \frac{\frac{\text{B}h}{\rho}}{\left(1 + \frac{h}{\rho}\right)} = \frac{\text{B}h}{\rho}\left(1 + \frac{h}{\rho}\right)^{-1}$$

Developing the last expression by the binominal formula

$$\text{B} - b = \frac{\text{B}h}{\rho} - \text{B}\,\frac{h^2}{\rho^2} + \text{B}\,\frac{h^3}{\rho^3}\ \text{\&c.}$$

Which is the correction, always subtractive.

But ρ being very great in comparison of h, it is sufficient for all ordinary values of B to take

$$\text{B} - b = \frac{\text{B}h}{\rho}$$

The horizontal alignment is made by means of a small transit, altitude, and azimuth instrument, or theodolite, and pickets with cards of pasteboard in their tops, placed along the line. The telescope of the instrument is directed to the distant extremity of the base, and then brought down on its horizontal or supporting axis, to indicate the proper position of each picket. The correctness of the ratio of the length of the arms of the vertical lever, to prevent changes in the distances between the contacts from changes of temperature, is tested by a pyrometer, invented by Mr. Saxton, of the Coast Survey. The rod is placed level upon two marble piers, sunk in the ground, which support its ends. The rod is then subjected to changes of temperature, artificially created, and being prevented from expanding in one direction by an upright stancheon, against which it abuts, its expansion, if any, in the opposite direction, acts against the vertical axis of a small plane mirror, giving the mirror an angular rotation about its vertical axis, the plane of the mirror always continuing vertical. At the opposite side of the room is a telescope placed horizontally on a pier, and directed to this mirror, and directly under the object glass of the telescope, is a horizontal scale of three feet in length, the divisions of which are $\frac{1}{4}$ of an inch. The movement of one of these divisions over the vertical wire of the telescope, occasioned by a motion of the small mirror in which it is reflected, corresponds to a change in length of the rod of $\frac{1}{25000}$ of an inch. The distance of the piers, which is liable to change, by hygrometric changes, is tested by a standard bar at the temperature of freezing. The standard bar is compared with a bar from France, the only one in the country. Its length is an exact French metre, which is the ten millionth part of a quarter of the meridian, the value of which in units already in use has been found by means of two measured degrees in distant latitudes. (See Appendix VI. p. 368.)

A bar of brass and iron exposed to the same temperature will not heat equally in equal times ; this is well known to depend upon the different conducting powers of the two metals, their different specific heats, and the different powers of their surfaces to absorb heat. The bars, then, if of equal sections, when the temperature is rising or falling, will not have the same temperature, and the system is not compensating. By adapting the sections according to rules deduced, partly by theory and partly by experiment, a small residual quantity remains to be corrected, which is detected only by the delicate tests of the Saxon pyrometer, or the lever of contact and level of Bessel. This correction is made by applying a covering more absorbent of heat to one bar than to the other.

The rods are surrounded by a tin covering, and of about twenty feet (six metres) in length, the whole being easily transported by four men.

TO REDUCE A BROKEN BASE TO A STRAIGHT LINE.

Let a and b be the given sides, and C the included angle, which is nearly 180°. Put $C = 180° - \theta$.* The formula is

$$c = a + b - \log.^{-1} (2 \cdot 6264222 + \log.\, a + \log.\, b + \text{ar. co. log. } (a + b) + 2 \log.\, \theta.)$$

The angles of the triangle of the great or primary chain are observed in England and this country with an instrument called

THE GREAT THEODOLITE.

That employed on the Coast Survey of the United States was made by Troughton and Simms, of London, under the direction of the late Mr. Hasler. It consists of a horizontal limb thirty inches in diameter, made narrow and light, with a vertical rim below, to strengthen it, supported by conical arms from beneath a central drum, upon which the telescope is mounted on pillars, as in the transit instrument. The telescope is of four feet focal length. The limb is graduated to 5′ spaces, and numbered to 360°. Three reading microscopes, reading to single seconds at intervals of 120°, are sustained above the limb, by horizontal arms projecting from the central drum. Three of the six horizontal arms which support the limb extend beyond the limb, and are provided with foot screws, upon which the instrument rests, on short pillars or legs, which fit each into one of three holes 120° apart, in a wooden frame strongly trussed.

The method of observing is as follows. The instrument is directed to the signal at one of the distant vertices of the triangle whose angles are to be measured, upon which the middle vertical wire is adjusted with great accuracy, and the reading taken by the three microscopes, the degrees for only one of them. The telescope is then inverted by turning it on its horizontal or supporting axis, and the instrument turned on its vertical axis 180°, till the middle vertical wire again coincides with the signal, when the reading of the three microscopes is again taken. The legs

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* θ being very small, the formula preceding (2) Art. 21, App. I., putting for $\sin \phi$ its value in (1), becomes, putting $\sin \frac{1}{2} \theta$ or $\frac{1}{2} \theta$ for its equal $\cos \frac{1}{2} C$, neglecting θ^4, &c.,

$$c = a + b - \tfrac{1}{2} \sin^2 1' \frac{ab\theta^2}{a + b}$$

$$= a + b - 0 \cdot 00000004231 \frac{ab}{a + b}\theta^2$$

Whence the formula in the text, in which θ is expressed in minutes en s[illegible]n the holes of the wooden frame, which changes the osition of the limb 120°, and the same operation gone through as efore, and so on.* As there are three holes for the legs, the whole number of readings on a single signal will thus amount to 18, the indiscriminate mean of which will involve a compensation for all instrumental rors.†

SELECTION OF STATIONS AND SIGNALS.

Eminences are usually chosen, and, if possible, such as to permit the ignals, which serve to mark the stations, being seen against the sky. A ery small object is thus made visible at a very great distance.

Permanent monuments of masonry or pottery are sunk below the surface of the ground far enough to be undisturbed by ploughing, with an [illegible] in the top, in which to insert a signal staff. The top of this staff [illegible]rts some object to which the telescope is directed. Spheres made [illegible] hoops, covered with white muslin, have been found to answer [illegible] ntainous regions, being visible fifty miles, but not so well [illegible] low grounds, and near the sea. Tin cones of an angle adapted to reflect advantageously the light of the morning and evening sun, were used by Mr. Hasler, who gives (Trans. Am. Phil. Soc., 1825), the mode of reducing in a simple manner the observation to the axis of the cone, a reduction depending on the relative position of the cone to the sun and the observer.‡ Steeples, circular or polygonal towers, windmills, &c., have been employed, and the methods of correcting the observations upon these are given by Puissant.§ At night, signals have been used formed of

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* This is for the purpose of measuring the same angle upon different parts of the limb.

† The French employ a large repeating circle, with which the angles are observed, in oblique planes, passing through the objects and the eye. These require to be reduced to the horizon by methods given by Puissant (Traité de Geodesie). The theodolites employed on the Coast Survey, besides the 30 inch, are one of 24 inches, and one of 18 inches, by Troughton and Simms, and a number of 14, 12, 10, and 6 inch repeating circles by Troughton and Simms, and Gambey, of Paris.

‡ A formula easily deducible from Hasler's is

$$\text{Correction in seconds} = \pm \frac{r \cos \frac{1}{2} z}{\text{D} \sin 1''}$$

In which r = mean radius of signal cone,

z = diff. of azimuth between ⊙ and signal,

D = distance.

§ Traité de Geodesie, a standard work on this subject, to which we shall have frequent occasion to refer.

lamps, placed at the focus of a parabolic reflector, or behind a lens at the focus of parallel rays. The most perfect is the Drummond light, which may be seen at the distance of seventy miles.

The best signal when the sun shines is one employed at present on the Coast Survey of the United States, called a *heliotrope*. It consists of a common telescope, mounted upon a three-legged stand, horizontally. It is accompanied by a man called a heliotroper, who directs the telescope to the tent in which the great theodolite is placed. Upon the eye end of the telescope is supported a small plane mirror, which has motion round both a vertical and horizontal axis, so as to be capable of being placed in a position to reflect the sun in any direction, at pleasure. The heliotroper attends and turns the mirror continually, so as to reflect the sun in a direction parallel to the axis of the telescope, which he accomplishes by causing the rays to pass through two perforated discs, supported like the mirror, on the top of the telescope tube, one being near the object end, and the other between it and the mirror.* The signal pole is supported by a wooden tripod at least $\frac{1}{8}$ its height, and the heliotrope is placed at a short distance from it, in a line with the theodolite station. All the signals visible from the station at which the great theodolite is placed, are observed every day for several weeks. The instrument is then moved to a new station, and by this means all the angles of every triangle are repeatedly observed.

REDUCTION TO THE CENTRE OF THE STATION.

It sometimes happens from the nature of the signal employed, that the theodolite cannot be placed at the axis of the signal called the *centre of the station*. The process of determining what the observed angle would have been with the instrument so placed, from observation with the instrument placed at a short measured distance† from the proper point, is called *reducing to the centre of the station.*

Let C in the diagram be the centre of the station, O the place of the instrument. From the observed angle AOB required the angle ACB. Make $AOB = \omega$, $BO = g$, $AO = d$, $ACB = x$, $OC = r$, $OOB = y$.

Then

$$AIB = \omega + IAO, \text{ and } AIB = x + CBO$$

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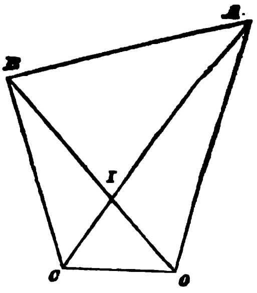
* The heliotroper is on duty till 10 A.M. and after 3 P.M., the atmosphere in the middle of the day being too unsteady near the earth's surface for good observations.

† If the centre of the station be inaccessible, this distance must be calculated from measurements which can be made by methods which the student will easily devise for any given case.



Equating these two values of AIB we obtain

$$x - \omega = \text{IAO} - \text{CBO}$$

But,

$$\sin \text{CAO} = \frac{r \sin(\omega + y)}{d}, \ \sin \text{CBO} = \frac{r \sin y}{g}$$

Substituting these values of the sines of CAO, CBO, for the angles themselves, which are very small, we have

$$x - \omega = \frac{r \sin(\omega + y)}{d \sin 1''} - \frac{r \sin y}{g \sin 1''},$$

which is the formula for the correction to be applied to the observed angle ω, to obtain the required angle x.

The distance r being small, d and g are computed with the angle ω.

VERIFICATION OF THE OBSERVED ANGLES.

The excess of the sum of the angles of a spherical triangle over two right angles is technically called the *spherical excess*. One third of the spherical excess being subtracted from each of the angles of a spherical triangle occupying but a small portion of the surface of the sphere, a plane triangle may be formed with the resulting angles, and with rectilinear sides, equal in length to the curvilinear sides of the spherical triangle, and A, B, C being the angles, s the area of the triangle, and r the rad. of the earth (or better the rad. of curvature, see App. VI. p. 366),

$$\text{A} + \text{B} + \text{C} = 180^\circ + \frac{s}{r^2 \sin 1''} \text{ *}$$

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* This is Legendre's theorem, the demonstration of which, by Lagrange, is as follows:—The sides of a spherical triangle being a, b, and c, and the radius of the sphere r, the similar triangle on the sphere, whose radius is 1, will have for sides $\frac{a}{r}, \frac{b}{r}, \frac{c}{r}$ which put equal to α, β, γ, and by Art. 82,

$$\cos \text{A} = \frac{\cos \alpha - \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \qquad (1)$$

But (p. 94) α and β being very small,

$$\cos \alpha = 1 - \frac{\alpha^2}{1 \cdot 2} + \frac{\alpha^4}{1 \cdot 2 \cdot 3 \cdot 4}, \ \sin \beta = \beta - \frac{\beta^3}{1 \cdot 2 \cdot 3}$$

and similar expressions obtain for $\cos \beta$, $\cos \gamma$, $\sin \gamma$ $\therefore$ (1) becomes which is the formula of verification, the last term being expressed in seconds.

If the sum of the three observed angles exceed 180° by $\frac{s}{r^2 \sin 1''}$ the observations are correct.

As the spherical excess $\frac{s}{r^2}$ is expressed in terms of the area of the triangle, that must be computed, which may be done with sufficient accuracy by considering the spherical triangle as plane, and applying to it one of the formulas of Art. 19, App. I.

The angles of the plane triangle, whose sides are equal in length to those of the given spherical triangle, being found by means of the spherical excess, as above explained, and the length of one of the sides of the

$$\cos \text{A} = \frac{\frac{1}{2}(\beta^2 + \gamma^2 - \alpha^2) + \frac{1}{24}(\alpha^4 - \beta^4 - \gamma^4) - \frac{1}{4}\beta^2\gamma^2}{\beta\gamma(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)} \qquad (2)$$

Transferring $(1 - \frac{1}{6}\beta^2 - \frac{1}{6}\gamma^2)$ to the numerator, by giving it the exponent -1, and developing as far as to terms of the fourth degree inclusive, (2) becomes

$$\cos \text{A} = \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} + \frac{\alpha^4 + \beta^4 + \gamma^4 - 2\alpha^2\beta^2 - 2\alpha^2\gamma^2 - 2\beta^2\gamma^2}{24\beta\gamma} \qquad (3)$$

Replacing the values of α, β, γ, (3) may be expressed as follows:—

$$\cos \text{A} = \frac{\text{M}}{2bc} + \frac{\text{N}}{24bcr^2} \qquad (4)$$

If the same lengths, a, b, c, be sides of a plane triangle, and A′ be the angle opposite a, by Art. 69,

$$\cos \text{A}' = \frac{b^2 + c^2 - a^2}{2bc} = \frac{\text{M}}{2bc} \qquad (5)$$

Raising both members of (5) to the square, and substituting $1 - \sin^2 \text{A}'$ for $\cos^2 \text{A}'$

$$-4b^2c^2\sin^2 \text{A}' = a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 = \text{N}$$

Equation (4) thus reduces to

$$\cos \text{A} = \cos \text{A}' - \frac{bc}{6r^2}\sin^2 \text{A}' \qquad (6)$$

Let $\text{A} = \text{A}' + x$, x being the difference between the spherical and plane angle,

$$\cos \text{A} = \cos \text{A}' \cos x - \sin \text{A}' \sin x = \cos \text{A}' \sqrt{1 - \sin^2 x} - \sin \text{A}' \sin x$$

Putting x, which is very small, for $\sin x$, and rejecting the second power of x,

$$\cos \text{A} = \cos \text{A}' - x \sin \text{A}' \qquad (7)$$

Combining (6) and (7), which have the same first members

$$x = \frac{bc}{6r^2}\sin \text{A}' \text{ and since } \text{A} = \text{A}' + x$$

spherical triangle being known, that of the others may be obtained by the solution of the plane triangle.

Between the latitudes 45° and 25° the spherical excess amounts to about 1″ for an area of 75·5 square miles. To find the spherical excess in seconds of space therefore divide the area in square miles by 75·5.

The logarithm of the mean radius of the earth in yards is 6·8427917, of r^2 is 13·6855834.

The following example shows the form* in which the above rules are applied in practice on the U. S. Coast Survey.

$$\text{A} = \text{A}' + \frac{bc}{6r^2} \sin \text{A}' \qquad (8)$$

But [Art. 19, (1) App. I.], $\frac{1}{2}\, b\, c$, $\sin \text{A}'$ is the area of the plane triangle of which a, b, c are the sides, which area does not differ sensibly from the proposed spherical triangle. If s denote the area of either of these (8) becomes

$$\text{A}' = \text{A} - \frac{s}{3r^2}$$

In a similar manner may be found

$$\text{B}' = \text{B} - \frac{s}{3r^2}$$

$$\text{C}' = \text{C} - \frac{s}{3r^2}$$

$$\therefore \text{A}' + \text{B}' + \text{C}' = \text{A} + \text{B} + \text{C} - \frac{s}{r^2} \qquad (9)$$

But $\text{A}' + \text{B}' + \text{C}' = 180^\circ \therefore \text{A} + \text{B} + \text{C} = 180^\circ + \frac{s}{r^2}$

And this last is the formula of verification. The same theorem has been extended to a spheroidal triangle, the difference between the spheroidal excess and spherical excess being less than $\frac{1}{20}$ of a second in the largest triangle ever measured on the surface of the earth. To express $\frac{s}{r^2}$ in seconds, it must be divided by sin 1″.†

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[*] In the form given the 1st, 3d, and 4th columns explain themselves; the 2d column contains the names of the stations at the vertices of the triangle, the 7th the spherical excess 5·67, calculated by the formula on p. 323, the 5th the difference 0·15 between the excess of the sum of the observed angles over 180°, and the spherical excess, which difference ought to be zero. This error 0·15 is equally distributed between the three angles; the 6th column contains the observed angles thus corrected; the 8th column contains the angles of a plane triangle whose sides are of the same length with the spherical one, each determined by subtracting ⅓ the spheri-

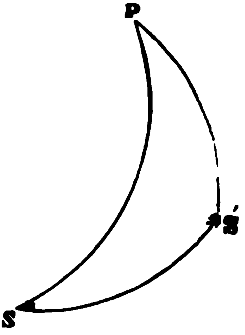
[†] A still more accurate formula, deduced rigorously from the spheroid, is $\text{Excess} = s\,\frac{1 + e^2 \cos 2\,\text{L}}{a^2}$, in which L = mean latitude, and a = equatorial radius. This form would only become important in carrying forward azimuths in a long chain.

No.	Denomination.	No. of Series.	Observed angles.	Distribution and Error.	Spherical angles.	Spherical excess.	Plane angles and distances.	Logarithms.
37	Manomet	36	66 34 04·80	—0″·05	04″·75	1·89	66 34 02·86	0·0373802
	Copecut	35	64 08 37·78	—0 ·05	37 ·73	1·89	64 08 35·84	9·9541886
	Bluehill	36	49 17 23·24	—0 ·05	23 ·19	1·89	49 17 21·30	9·8796760
				—0 ·15		5·67		
				Bluehill to Copecut,			54695·61	4·7379524
				Manomet to Bluehill,			53644·00	4·7295212
				Manomet to Copecut,			45186·49	4·6550086

DETERMINATION OF THE LATITUDES, LONGITUDES AND AZIMUTHS OF THE STATIONS.

The latitude and longitude of a few of the stations of a chain of triangles, selected at different parts of the whole, being determined by astronomical observations, the latitudes and longitudes of the other stations may be found *geodetically*, as it is termed, by methods which we now proceed to explain.

The problem to resolve is the following. Given the latitude and the longitude of the point S in the diagram, and the azimuth of the point S′ upon the horizon of S*, to find the latitude and longitude of this second point, and the azimuth of S upon the horizon of S′.



If L denote the given latitude, dL the difference of latitude between the two stations to be applied as a correction to the known latitude, in order to find the required latitude, M the longitude, dM the difference of longitude of the two stations, Z the azimuth, dZ the difference of azimuth; then the following will be the

FORMULAS FOR COMPUTATION OF L, M, Z OF PRIMARY TRIANGLES.

For the difference of latitude

$$-d\text{L} = \text{KB}\cos\text{Z} + \text{K}^2\,\text{C}\,\sin^2\text{Z} + (d\text{L})^2\,\text{D} - h\,\text{K}^2\,\text{E}\,\sin^2\text{Z}\dagger$$

---

cal excess, or 1·89 from the corresponding spherical angles; this column also contains the sides of the plane triangle, the last two of which are computed from the first, and the angles by the sine proportion, p. 62, the logarithms for which purpose are contained in the last column.

* Observed astronomically by methods to be described in the sequel.

† The following is the demonstration of this formula:—There are known in the triangle PSS′ the colatitude PS = 90° — L of the point S, the angle PSS′ = the azimuth

In which K = dist. between the two stations (in metres)

$$\mathrm{B} = \frac{1}{\mathrm{R}\ \mathrm{arc}\ 1''} = \frac{(1 - e^2 \sin^2 \mathrm{L})^{\frac{3}{2}}}{a\,(1 - e^2) \sin 1''} \text{ since } \mathrm{R}^{*} \ (= \text{rad. of curvature}) = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \mathrm{L})^{\frac{3}{2}}}$$

of S' from $\mathrm{S} = \mathrm{Z}$, and the side $\mathrm{SS}' = \mathrm{K}$, the geodetic line or measured distance between S and S'. That is to say two sides and the included angle, to compute PS', the colatitude, and $\mathrm{PS}'\mathrm{S} = 180^\circ - \mathrm{Z}'$ the azimuth of S from S', and $\mathrm{P} = d\mathrm{M}$, the diff. of long. required.

The determination of the latitude, especially, by this method, would be attended with considerable error. We shall deduce the formula given by Delambre, for the difference of latitude between the two stations. The formula of Art. 82, applied to the present triangle, gives

$$\cos \mathrm{PS}' = \cos \mathrm{S} \sin \mathrm{PS} \sin \mathrm{SS}' + \cos \mathrm{PS} \cos \mathrm{SS}'$$

or, since $\mathrm{L} + d\mathrm{L}$ = the latitude of S', of which PS' is the complement

$$\sin (\mathrm{L} + d\mathrm{L}) = \cos \mathrm{Z} \cos \mathrm{L} \sin \mathrm{K} + \sin \mathrm{L} \cos \mathrm{K}$$

or [Art. 70, (3)];

$$\sin \mathrm{L} \cos d\mathrm{L} + \cos \mathrm{L} \sin d\mathrm{L} = \cos \mathrm{Z} \cos \mathrm{L} \sin \mathrm{K} + \sin \mathrm{L} \cos \mathrm{K}$$

or [Art. 72 (8), and Art. 71 (3)],

$$\sin \mathrm{L}\,(1 - 2 \sin^2 \tfrac{1}{2} d\mathrm{L}) + 2 \cos \mathrm{L} \cos \tfrac{1}{2} d\mathrm{L} \sin \tfrac{1}{2} d\mathrm{L} = \cos \mathrm{Z} \cos \mathrm{L} \sin \mathrm{K} + \sin \mathrm{L}\,(1 - 2 \sin^2 \tfrac{1}{2} \mathrm{K})$$

Whence, reducing

$$-2 \sin \mathrm{L} \sin^2 \tfrac{1}{2} d\mathrm{L} + 2 \cos \mathrm{L} \cos \tfrac{1}{2} d\mathrm{L} \sin \tfrac{1}{2} d\mathrm{L} = \cos \mathrm{Z} \cos \mathrm{L} \sin \mathrm{K} - 2 \sin \mathrm{L} \sin^2 \tfrac{1}{2} \mathrm{K}$$

Dividing by 2 cos L this becomes

$$-\tan \mathrm{L} \sin^2 \tfrac{1}{2} d\mathrm{L} + \cos \tfrac{1}{2} d\mathrm{L} \sin \tfrac{1}{2} d\mathrm{L} = \tfrac{1}{2} (\cos \mathrm{Z} \sin \mathrm{K} - 2 \tan \mathrm{L} \sin^2 \tfrac{1}{2} \mathrm{K})$$

Representing the second member by p, and $-\tan \mathrm{L}$ by q, and dividing all the terms by $\cos^2 \frac{1}{2} d\mathrm{L}$, we have

$$q \tan^2 \tfrac{1}{2} d\mathrm{L} + \tan \tfrac{1}{2} d\mathrm{L} = \frac{p}{\cos^2 \frac{1}{2} d\mathrm{L}} = p\,(1 + \tan^2 \tfrac{1}{2} d\mathrm{L})$$

or,

$$(q - p) \tan^2 \tfrac{1}{2} d\mathrm{L} + \tan \tfrac{1}{2} d\mathrm{L} = p$$

Resolving this quadratic we have

$$\tan \tfrac{1}{2} d\mathrm{L} = -\frac{1}{2\,(q - p)} + \frac{1}{2\,(q - p)} [1 + 4\,p\,(q - p)]^{\frac{1}{2}}$$

Developing the radical part by the binomial theorem,

$$\tan \tfrac{1}{2} d\mathrm{L} = \frac{1}{2\,(q - p)} [2\,(q - p)\,p - 2\,(q - p)^2 p^2 + 4 \quad (q - p]^3 p^3 \text{ \&c.}]$$

Or performing the operation indicated in the second member, and rejecting the terms beyond p^3,

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[*] Values of R and N for every latitude are furnished by a table, p. 44, Part I., of Lee's Tables and Formulas. See also (4), p. 366 and p 368 App. VI.

$$\mathrm{C} = \frac{\tan \mathrm{L}}{2\,\mathrm{N}\,\mathrm{R}\,\mathrm{arc}\,1''} = \frac{\tan \mathrm{L}\,(1 - e^2 \sin^2 \mathrm{L})^2}{a^2\,(1 - e^2)\,2 \sin 1''}, \mathrm{N} \text{ being} = \frac{a}{(1 - e^2 \sin^2 \mathrm{L})^{\frac{1}{2}}}$$

$$\mathrm{D} = \frac{\frac{3}{2}\,e^2 \sin \mathrm{L} \cos \mathrm{L}\,\mathrm{arc}\,1''}{(1 - e^2 \sin^2 \mathrm{L})^{\frac{3}{2}}}, \mathrm{E} = \frac{1 + 3 \tan^2 \mathrm{L}}{6\,\mathrm{N}^2} = \frac{(1 + 3 \tan^2 \mathrm{L})\,(1 - e^2 \sin^2 \mathrm{L})}{6\,a^2}$$

$$\tan \tfrac{1}{2}\,d\mathrm{L} = p - qp^2 + (1 + 2q^2)\,p^3$$

But (see note p. 95),

$$\tfrac{1}{2}\,d\mathrm{L} = \tan \tfrac{1}{2}\,d\mathrm{L} - \tfrac{1}{3} \tan^3 \tfrac{1}{2}\,d\mathrm{L}$$

Substituting in the second member of this the value of $\tan \frac{1}{2}\,d\mathrm{L}$, given by the preceding it becomes

$$\tfrac{1}{2}\,d\mathrm{L} = p - q\,p^2 + (1 + 2q^2)\,p^3 - \tfrac{1}{3}\,p^3$$

rejecting the higher powers of p. Or,

$$d\mathrm{L} = 2p - 2qp^2 + \tfrac{4}{3}\,p^3\,(1 + 3q^2)$$

But;

$2\,p = \cos \mathrm{Z} \sin \mathrm{K} - 2 \tan \mathrm{L} \sin^2 \frac{1}{2}\,\mathrm{K}$

$2\,q\,p^2 = -\frac{1}{2} \cos^2 \mathrm{Z} \sin^2 \mathrm{K} \tan \mathrm{L} + 2 \cos \mathrm{Z} \sin \mathrm{K} \sin^2 \frac{1}{2}\,\mathrm{K} \tan^2 \mathrm{L}$

$\frac{4}{3}\,p^3 = \frac{1}{6} \cos^3 \mathrm{Z} \sin^3 \mathrm{K}$, rejecting terms containing higher powers of $\sin \mathrm{K}$ than the third.

But [App. I., p. 94 (1)],

$$\sin \mathrm{K} = \mathrm{K} - \tfrac{1}{6}\,\mathrm{K}^3$$

Whence the first term in the value of $2p$ becomes

$$\mathrm{K} \cos \mathrm{Z} - \tfrac{1}{6}\,\mathrm{K}^3 \cos \mathrm{Z} \qquad (m)$$

and the value of $\frac{4}{3}\,p^3$ may be put under the form

$\frac{1}{6}\,\mathrm{K}^3 \cos^2 \mathrm{Z} \cos \mathrm{Z}$, rejecting higher powers of K than K^3

This last, added to the second term of (m), gives

$$-\tfrac{1}{6}\,\mathrm{K}^3 \cos \mathrm{Z}\,(1 - \cos^2 \mathrm{Z}) = -\tfrac{1}{6}\,\mathrm{K}^3 \cos \mathrm{Z} \sin^2 \mathrm{Z}$$

so that (m) thus increased becomes

$$\mathrm{K} \cos \mathrm{Z} - \tfrac{1}{6}\,\mathrm{K}^3 \sin^2 \mathrm{Z} \cos \mathrm{Z}$$

Again, putting $\frac{1}{2}\,\mathrm{K}$ for its sine, being a small arc, the second term in the value of $2p$ becomes

$$-\tfrac{1}{2}\,\mathrm{K}^2 \tan \mathrm{L}$$

and the first term of the value of $-2qp^2$ becomes

$$+\tfrac{1}{2}\,\mathrm{K}^2 \cos^2 \mathrm{Z} \tan \mathrm{L}$$

These last two being added together produce

$$-\tfrac{1}{2}\,\mathrm{K}^2 \tan \mathrm{L} \sin^2 \mathrm{Z}$$

Thus the expression for the difference of latitude reduces to

$$d\mathrm{L} = \mathrm{K} \cos \mathrm{Z} - \tfrac{1}{2}\,\mathrm{K}^2 \sin^2 \mathrm{Z} \tan \mathrm{L} - \tfrac{1}{6}\,\mathrm{K}^3 \sin^2 \mathrm{Z} \cos \mathrm{Z}\,(1 + 3 \tan^2 \mathrm{L}) \text{ \&c.}$$

Delambre observes that the meridional arcs, computed on the supposition that the earth is a sphere differ insensibly from those computed on the spheroidal hypomidiameter of the earth = 6377397·15 metres, log. a = 6·8046434637, h = 1st term K·B cos Z.

$e = 0{\cdot}0816967$, log. $e = 8{\cdot}9122052271$.

Subjoined is part of a table, from which B, C, D, E are to be taken.

Argum. Latitude.	Log. B.	Diff. 10″	Log. C.	Diff. 10″	Log. D.	Log. E.	Diff. 10″
		—		+			+
40° 00′	8·5108910·6		1·32843·1		2·3801·9	6·1044·4	
1	898·1		868·6		02·6	48·6	
2	885·6		894·1		03·4	52·7	
3	873·1		919·6		04·1	56·9	
4	860·6		945·1		04·9	61·0	
5	848·1	2·08	970·5	4·25	05·6	65·2	0·69
6	835·6		996·0		06·4	69·4	
7	823·1		1·33021·5		07·1	73·5	
8	810·6		047·0		07·9	77·7	
9	798·1		072·5		08·6	81.8	
10	8·5108785·6		1·33098·0		2·3809·4	6·1086·0	

For an example, let us make K = 100 miles = 160900 metres, L =

thesis. The consideration of the spheroidal figure is only of consequence, in converting the terrestrial arcs, measured in metres, into seconds. This may be done by dividing such arcs by their radii of curvature,[1] and this quotient by the length of 1″. Our formula is deduced by considering the normal N terminating at the polar axis as the radius of curvature of the geodetic arc K, which it is very nearly; then if K and N be both expressed in metres,

$$\frac{K}{N}$$

will be the length of K in terms of N as unity. If therefore

$$\frac{K}{N}$$

be substituted for K in the last value of dL, we have

$$dL = \frac{K \cos Z}{N} - \frac{1}{2}\,\frac{K^2 \sin^2 Z \tan L}{N^2} - \frac{1}{6}\,\frac{K^3 \sin^2 Z \cos Z\ (1 + 3 \tan^2 L)}{N^3} \qquad (g)$$

in which dL is expressed in terms of N the radius of K as unity.

In order that it may be expressed in terms of its own radius as unity, which we will call R,[2] this value of dL must be multiplied by

$$\frac{N}{R}$$

and if the result thus obtained be divided by the length of 1″ in terms of radius unity, the value of dL will be expressed in seconds.

Multiplying therefore the second member of (g) throughout by $\frac{N}{R}$ dividing it by arc 1″, and estimating Z from S round by W, from 0° to 360°, which will be the supplement of the Z used in the diagram, it becomes

$$-\,dL = +\,\frac{K \cos Z}{R \text{ arc } 1''} + \frac{K^2 \sin^2 Z \tan L}{2NR \text{ arc } 1''} - \frac{K^3 \sin^2 Z \cos Z}{6\,N^2\,R \text{ arc } 1''}\,(1 + 3 \tan^2 L) \qquad (a)$$

---

[1] The radius of curvature at any point of a curve is the radius of a circle having the same curvature as the curve in that part of it.

[2] For the demonstration of a formula for the radius of curvature of an elliptical meridian, at any point of that meridian, in terms of the latitude of that point, see the Appendix to Part VI. p. 366.

40°, Z = 54°; the computation by means of the preceding table is as follows:

K log.	5·2065560	K² log.	10·41311		
Z log. cos	9·7692187	Z log. sin²	9·81592	$(d\text{L})^2$* log.	6·9835
B log.	8·5108911	C log.	1·32843	D log.	2·3802
α log.	3·4866658	β log.	1·55746		9·3637

h log.	3·4867	1st term	3066″·661
K² sin² Z log.	0·2290	2d "	+ 36 ·096
E log	6·1044	3d "	0 ·231
	9·8201		3102 ·988
		4th "	0 ·661
			3102·327

The 1st and 2d terms should always be used. The third term should be used whenever log. α is over 2·310. When K is less than 100,000 metres, or log. K not over 5·00, α^2 may be written in the 3d term for $(d\text{L})^2$ for all Z° less than 35°, or wherever β is not over 0·876.

$$\text{where R} = \frac{a\,(1 - e^2)}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}}{}^{1} \quad \text{N} = \frac{a}{(1 - e^2 \sin^2 \text{L})^{\frac{1}{2}}}$$

The results are better if the radius of curvature midway between the two latitudes is used; make $\frac{1}{2}\, d\text{L} = \lambda$; $\text{L} - \lambda = \text{L}_m$ the latitude of the middle point between L and L′, and the radius of curvature at $\text{L}_m = \text{R}_m$; then should the formula (a) be multiplied by $\frac{\text{R}}{\text{R}_m}$ if we would refer the arc $d\text{L}$ to R_m. Then $d\text{L}\,\frac{\text{R}}{\text{R}_m} - d\text{L} = \frac{\text{R} - \text{R}_m}{\text{R}_m}\, d\text{L}$ is the correction to be added to (a) to make the required reduction.

$$\text{Now is R} - \text{R}_m = a\,(1 - e^2)\left[\frac{1}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} - \frac{1}{(1 - e^2 \sin^2 \text{L}_m)^{\frac{3}{2}}}\right] = a\,(1 - e^2)$$

$$\frac{1 - \frac{3}{2}\, e^2 \sin^2 \text{L}_m - 1 + \frac{3}{2}\, e^2 \sin^2 \text{L}}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}\,(1 - e^2 \sin^2 \text{L}_m)^{\frac{3}{2}}} = a\,(1 - e^2)\,\frac{\frac{3}{2}\, e^2 (\sin^2 \text{L} - \sin^2 \text{L}_m)}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}\,(1 - e^2 \sin^2 \text{L}_m)^{\frac{3}{2}}}$$

But since (See. 63 p. 103), $\sin^2 \text{L} - \sin^2 \text{L}_m = \sin(\text{L} - \text{L}_m)\sin(\text{L} + \text{L}_m) = \sin\lambda \sin(2\,\text{L} - \lambda) = d\text{L} \sin \text{L} \cos \text{L}$ very nearly, because $\frac{1}{2}\sin 2\,\text{L} = \sin \text{L} \cos \text{L}$. The quantity neglected here is quite insensible in practice.

$$\text{We have then } \frac{\text{R} - \text{R}_m}{\text{R}_m} = \frac{a\,(1 - e^2)\,\frac{3}{2}\, e^2\, d\text{L} \sin \text{L} \cos \text{L}}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}\,(1 - e^2 \sin^2 \text{L}_m)^{\frac{3}{2}}} \times \frac{(1 - e^2 \sin^2 \text{L}_m)^{\frac{3}{2}}}{a\,(1 - e^2)}$$

$$= \frac{\frac{3}{2}\, e^2\, d\text{L} \sin \text{L} \cos \text{L}}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} \text{ and } \frac{\text{R} - \text{R}_m}{\text{R}_m}\, d\text{L} = (d\text{L})^2\, \frac{\frac{3}{2}\, e^2 \sin \text{L} \cos \text{L}}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} \qquad \text{(c)}$$

As this quantity never amounts to $\frac{1}{10000}$ of $d\text{L}$, the third term of $d\text{L}$ in (a) need not

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* This $d\text{L}$ is the sum of the first two terms $a + \beta$.

[1] See Appendix VI. p. 366.

For secondary work the 4th term $h\text{K}^2 \sin^2 \text{Z}.\text{E}$ may always be omitted. The 3d term very frequently is of no sensible value, and a^2 may *always* be written in the place of $d\text{L}^2$, when K does not exceed 86·500 metres (54 miles), or log. K = 4·937, which comprise the vast majority of cases. When K is less than 34000 metres, two terms are sufficient. The best rule for the omission of the third term is that it need not be used unless log. a is greater than 2·31 . . ., log. D (which scarcely varies), being about 2·38 . . ., we shall, in that case, have log a^2 D = 7·00 . . = log. of 0″·001.

It appears that there are hardly any cases in which the second term may be omitted.

The 1st term gives the distance on the meridian of the point of departure from that point to the foot of the perpendicular from the second point, the second term gives the reduction to the parallel. It is only at a very small azimuth then, that the 2d term may be neglected even for a very short line.

The 4th term may be omitted between latitude 45° and 40°, when K is not over 17000 metres, or log. K = 4·2304. Between latitude 40° and 35°, when K is not over 18500 metres, or log. K = 4·2671, and between latitude 35° and 30°, when K is less than 20,000 metres, or log. K 4·301.

In computing carry log. B to 7 places,
" " " C to 5 places.
" " " D and E to 4 places.

The formula for difference of longitude is

$$d\text{M} = \frac{\text{A}'\text{K} \sin \text{Z}}{\cos \text{L}'} {}^{*} \qquad \text{(G)}$$

be used in making the substitution of $d\text{L}$ in (c), and the second term only when it is over 18°. If in (c) we introduce $d\text{L}$ expressed in seconds obtained by (a), we must of course multiply by arc 1″, and we have, finally,

$$-d\text{L} = \frac{\text{K} \cos \text{Z}}{\text{R} \operatorname{arc} 1''} + \frac{\text{K}^2 \sin^2 \text{Z} \tan \text{L}}{2\,\text{N}\,\text{R} \operatorname{arc} 1''} - \frac{\text{K}^3 \sin^2 \text{Z} \cos \text{Z}}{6\,\text{N}^2\,\text{R} \operatorname{arc} 1''}(1 + 3 \tan^2 \text{L}) + (d\text{L}^2)$$

$$\frac{\frac{3}{2} e^2 \sin \text{L} \cos \text{L} \operatorname{arc} 1''}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} \qquad \text{(A)}$$

Making $\frac{1}{\text{R} \operatorname{arc} 1''} = \text{B}$, $\frac{\tan \text{L}}{2\,\text{NR} \operatorname{arc} 1''} = \text{C}$, $\frac{\frac{3}{2} e^2 \sin \text{L} \cos \text{L} \operatorname{arc} 1''}{(1 - e^2 \sin^2 \text{L})^{\frac{3}{2}}} = \text{D}$, $\frac{1 + 3 \tan^2 \text{L}}{6\,\text{N}^2} = \text{E}$, and call the first term h, we may write $-d\text{L} = + \text{K} \cos \text{Z}\cdot\text{B} + \text{K}^2 \sin^2 \text{Z}\cdot\text{C} + (d\text{L})^2 \text{D} - h\,\text{K}^2 \sin^2 \text{Z}\cdot\text{E}$, in which E may be taken out at the same time and from the same page as B and C; h could be copied from the bottom, or sum of the logs. of the 1st term, and $\text{K}^2 \sin^2 \text{Z}$, by taking the sum of the first two logs. of the 2d term.

[*] The formula for the difference of longitude is obtained in an obvious manner, by applying the sine proportion (Art. 81), to the spherical triangle PSS′, which gives, writing $d\text{M}$ for sin $d\text{M}$ = sin P and K for sin K,

In which

$$\mathrm{A}' = \frac{1}{\mathrm{N}\ \mathrm{arc}\ 1''} = \frac{(1 - e^2 \sin^2 \mathrm{L}')\ \frac{1}{2}}{a\ \sin\ 1''}$$

L' = new latitude, computed from the formula for $-d\mathrm{L}$.

$$d\mathrm{M} = \frac{\mathrm{K} \sin \mathrm{Z}}{\cos \mathrm{L}'},$$

And the value of $d\mathrm{M}$ in seconds of arc is obtained by converting K into seconds, by dividing K in metres by $\mathrm{N} \sin 1''$, N being the normal, and the length of the radius used at that part of the earth in metres. The above formula thus becomes the one already given (G). Lee's tables and formulæ gives a table for log. N, log. $\frac{1}{\mathrm{N} \sin 1''}$ and log. $(1 + e^2 \cos^2 \mathrm{L})$ for any latitude between 20 and 50 degrees.

(G) is founded on the supposition that $\cos \mathrm{L}' : \sin \mathrm{Z} :: \mathrm{K} : d\mathrm{M}$, whereas, in reality, $\cos \mathrm{L}' : \sin \mathrm{Z} :: \sin \frac{\mathrm{K}}{\mathrm{N}} : \sin d\mathrm{M}$. The error committed by the former supposition is expressed by $\frac{\mathrm{K}^3}{\mathrm{N}^3\ 6 \cos \mathrm{L}'\ \mathrm{arc}\ 1''} \left(\frac{\sin^3 \mathrm{Z}}{\cos^2 \mathrm{L}'}, - \sin \mathrm{Z} \right)$ [see (1) p. 94], or for $\mathrm{L}' = 45^\circ$, $\frac{\mathrm{K}^3}{3 \sqrt{2}\ \ \mathrm{N}^3\ \mathrm{arc}\ 1''} (2 \sin^3 \mathrm{Z} - \sin \mathrm{Z})$. This is a maximum when $\mathrm{Z} = 24^\circ\ 06'$, and if we substitute this Z in the latter expression, and make it equal to 0·001″, we find the corresponding log. K to be 4·4315 = log. of 27000. For any line over 27000 metres, then a correction ought to be applied to $d\mathrm{M}$, or if we will allow an error of 0·002, for any line over 34000 metres = about 21 miles.

In the annexed table, the column headed $d\mathrm{M}$ contains the log. of the seconds in a given arc; the column headed diff. contains the diff. between the log. of that arc and the log. of its sine (to the seventh place); the column headed K containing the log. of the length of that arc in metres. To apply the correction in question after having first computed $d\mathrm{M}$ by the formula (G), enter the table with the given log. K, and take out the corresponding diff.; again enter the table with the computed log. $d\mathrm{M}$, and take out the corresponding diff., and lastly, subtract the difference between the two quantities thus obtained from log. $d\mathrm{M}$, the result will be the corrected log. $d\mathrm{M}$.[1]

K	diff.	$d\mathrm{M}$	diff.
contains log. K in metres.	contains diff. between log. arc and log. sin to seven places of decimals.	contains log. $d\mathrm{M}$ in seconds of arc.	contains diff. between log. of arc and log. sine to seven places of decimals.

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[1] For denoting the difference between the log. arc and log. sin by δ, the formula (G) should be after the application of logarithms,

$$\log.\ d\mathrm{M} + \delta \log.\ d\mathrm{M} = \log. \frac{\mathrm{K}}{\mathrm{N}} + \delta \log. \frac{\mathrm{K}}{\mathrm{N}} + \log. \frac{\sin \mathrm{Z}}{\cos \mathrm{L}}.$$ Whence the rule is obvious.

Log. A′ s be carried to eight places. Seven places of logarithms should be used for $d\text{M}$.

For azimuth we have

$$-dz = \frac{d\text{M} \sin \lambda}{\cos \frac{1}{2}\, d\text{L}} \qquad (\text{D})^*$$

$$\lambda = \tfrac{1}{2}(\text{L} + \text{L}')$$

$$z' = 180^\circ + z - \frac{d\text{M} \sin \lambda}{\cos \frac{1}{2}\, d\text{L}}$$

For any line less than 340000 metres (21 miles) $\cos \frac{1}{2}\, d\text{L}$ may be omitted, being regarded as 1.

In computing dz, $\sin [\lambda = \frac{1}{2}(\text{L} + \text{L}')]$ is taken out to five places for main chain of triangles, and to four for the others, carrying forward dz in tenths of seconds in the first, and in whole seconds in the second.

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* The formula for the difference of azimuth is deduced as follows:—In the tri-e PSS′ we have, by Napier's analogies, calling π, π' the polar distance of S, S′,

$$\tan \tfrac{1}{2}(s' + s) = \frac{\cot \frac{1}{2} \text{P} \cos \frac{1}{2}(\pi - \pi')}{\cos \frac{1}{2}(\pi + \pi')}$$

Or recollecting that $\tan = \frac{1}{\cot}$ (Art. 37).

$$\cot \tfrac{1}{2}(s' + s) = \frac{\tan \frac{1}{2} \text{P} \cos \frac{1}{2}(\pi + \pi')}{\text{os} \frac{1}{2}(\pi - \pi')} = \frac{\tan \frac{1}{2}\, d\text{M} \sin \frac{1}{2}(\text{L} + \text{L}')}{\cos \frac{1}{2}\, d\text{L}} \qquad (n)$$

But $s' = 180^\circ - z'$, and $s = z$, and $\cot \frac{1}{2}(s' + s) = \tan [90^\circ - \frac{1}{2}(s' + s)]$ $\therefore$ n becomes

$$\tan \tfrac{1}{2}(z' - z) = \frac{\tan \frac{1}{2}\, d\text{M} \sin \lambda}{\cos \frac{1}{2}\, d\text{L}}$$

which is the formula (D) above, if we write $\frac{1}{2}\, dz$, for $\tan \frac{1}{2}(z' - z)$ and $\frac{1}{2}\, d\text{M}$ for $\tan \frac{1}{2}\, d\text{M}$.

The formula for dz requires some amendment within the same limits, within which we obtain $d\text{L}$ and $d\text{M}$; we have

$$\tan \tfrac{1}{2}(z' - z) = \tan \tfrac{1}{2}\, d\text{M} \frac{\sin \lambda}{\cos \frac{1}{2}\, d\text{L}} \qquad (1)$$

for which we have hitherto used

$$dz = d\text{M} \frac{\sin \lambda}{\cos \frac{1}{2}\, d\text{L}}$$

By transformation of (1) we get (see note p. 94, and make $\cos^2$ and $\cos^3$ of $\frac{1}{2}\, d\text{L} = 1$)

$$dz = d\text{M} \frac{\sin \lambda}{\cos \frac{1}{2}\, d\text{L}} + \tfrac{1}{12}\, d\text{M}^3 \sin \lambda \cos^2 \lambda \sin^2 1''$$

We write the second term thus, $+\, d\text{M}^3\, \text{F}$ where log. F is to be taken from the tables, into which it can easily be inserted, as only one value will be required for every half degree of L. It is 7·8324 for 25°, and 7·8404 for 45°; diff. for 30′ = 0·0002. The term $d\text{M}^3\, \text{F}$ can never exceed 0″·1.

Whenever the log. of any term is not over 7·00 . . the corresponding number need not be taken out.

Azimuths are reckoned from south round by west, and from 0° to 360°, the signs of sin z and cos z varying accordingly.

The following form, filled up with an example, is that at present used on the Coast Survey of the United States for the computation of the above formulæ for difference of latitude, longitude, and azimuth.

				°	′	″
Z	A	to	B	101	23	16·757
∠	B	and	C	83	36	43·243
Z	A	to	C	185	00	00·000
−dz				+	05	47·413
180						
z′	C	to	A	5	05	47·413

	°	′	″			°	′	″
L	45	00	00·000	A	M	70	31	50·000
−dL	1	04	32·264		d M	—	08	06·748
L′	46	04	32·264	C	M′	70	23	43·252

K	5·0791812·5	K^2	0·15836			h	3·5880
cos z	9·9983442·2	$\sin^2 z$	7·88059	δL^2	7·1760	$K^2 \sin^2 z$	8·0389
B	8·5105124·1	C	1·40411	D	2·3872	E	6·2132
h	3·5880379		9·44306		9·5632		7·8401

1st term	3872·914 —	3d term	0·366				
2d term	0·277 +	4th term	0·007				
	3872·637 —				—249		
3d & 4th terms	0·373 +			A′	8·5090285	Arg.	
−dL	3872·264 —	d M	2·6873039	K	5·0791812	K	—255
λ	45° 32′ 16″	sin λ	9·8535235	sin z	8·9402960	d M	+ 6
½dL	32 16	cos ½dL ar. co.	191	cos L′ ar. co.	0·1588231		—249
			2·5408465		2·6873039		
		−d z	347″·413	d M	486″·748		

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[*] In this form the first horizontal line expresses the azimuth of the line joining the two stations A and B; the second the angle formed by a line from A to a third station C, with AB; the third is found by the addition of these, and is the azimuth of AC; the fourth the excess of the difference of azimuth between AC and CA over 180°, computed below at the bottom of the form; the fifth the azimuth of CA required, formed by the addition of the two above, and 180°. The sixth horizontal line contains the latitude L and longitude M of A; the seventh the difference of latitude and longitude of A

PROJECTION OF MAPS.

The geographical positions of the vertices of the triangles having been determined by calculation, as above explained, blank maps are prepared with lines upon them representing meridians and parallels of latitude, upon which these points are accurately put down in their true positions, and the maps thus prepared being placed in the hands of the plane table parties, are filled up with the details of the ground which they represent, in the manner described at p. 235 et seq., the points marked upon them, and identified upon the ground by the sunk masonry or pottery of the signals employed in the triangulation becoming the base points or points of departure for the operations of the plane table.

The mode of preparing these maps in practice, it will be now proper to explain.

A spherical or spheroidal surface like that of the earth not being developable, it is impossible to represent upon a plane the positions of places without changing more or less their distances from one another.* When a small portion of the earth's surface is to be represented, the best mode is to conceive the earth to be enveloped by a tangent cone, the

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and C, computed below; the eighth the latitude L' and longitude M' of C, found by taking the algebraic sum of the two above. The next four horizontal lines of the form contain the computation of the difference of latitude dL between A and C, the first column being the computation of the logarithm of the first term, the second that of the second term, and so on, of the formula dL, at p. 325. The next two lines contain the four terms themselves,[1] and the next two their combination to form $-dL$. The first two columns of the remainder of the form contain the computation of $-dz$; the third column that of the difference of longitude of A and C, viz. dM, and the fourth the correction of this, which is sometimes employed. Applied here the log. dM becomes 2·6873039, and dM 486·748, differing only 0''·012 from what it was without the correction. A correction is also sometimes applied to dz, as has been already stated at p. 332, the formula for which is $F dM^3$, in which $F = \frac{1}{12} \sin\lambda \cos^2\lambda \sin^2 1''$. The computation of this correction in the present example would be as follows:—

$$\begin{array}{r} dM^3 = 8{\cdot}0619 \\ F = 7{\cdot}8404 \\ \hline 5{\cdot}9023 \end{array}$$

In which F may be taken from a table previously prepared. The last number is the logarithm of the correction to be applied to $-dz$.

* For the ordinary modes of projecting the hemisphere, see "spherical projections" in Davies' Descriptive Geometry, and for the analytical investigations of the same, see Francœur (Geodésie, 209, et seq.)

[1] z being between 90° and 270°, cos z is negative, and ∴ h is negative.

circle of contact being the middle parallel of the region to be embraced, and to suppose the surface of this cone to coincide with that of the earth over the whole extent between the northern and southern parallels of the map. This cone, when developed, becomes the sector of a circle, the portion of which between the two extreme parallels which we have supposed to embrace the surface of contact, will represent the surface of the map.

Supposing the earth to be spherical, which may always be done in the projection of maps, its oblateness being so small, and representing the latitude of the middle parallel of the map by λ, and the number of degrees of longitude to be contained in the map by D, it is evident that the absolute length of the middle parallel of the map will be expressed by (see 1st note, p. 153)

$$\frac{D}{180^\circ}\,\pi \cos \lambda \qquad (1)$$

In the above expression the radius of the earth is unity, and this being the case, the slant height or length of the element of the cone from the vertex to the circle of contact will evidently be the cotangent of the latitude. The arc of the sector, which is expressed by (1), divided by its radius cot λ, gives the length of the arc which measures the angle of the sector to radius unity. The result is

$$\frac{D}{180^\circ}\,\pi \sin \lambda$$

and this, which is the absolute length of the arc, must be multiplied by $\frac{180^\circ}{\pi}$* to have its value, or the measure of the angle of the sector in degrees,

$$D \sin \lambda \qquad (2)$$

then will be the formula for this angle, and the construction of the map will be very simple. It will only be necessary to draw two lines forming the angle expressed by (2), and with a radius equal to cos λ and the vertex of the angle as a centre, the arc representing the mean parallel may be described. If the map is to contain d degrees of latitude, then $\frac{\pi d}{180^\circ}$ will express the distance between the extreme parallels, and by describing arcs from the vertex of the sector with radii greater and less than

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* $\frac{180^\circ}{\pi} = 59^\circ,29573 = 3437',74677 = 206264'',80625.$

λ by th f of this expression, the extreme parallels of the map will be constructed. The distance between the parallels is then divided into any number of equal parts at pleasure, and arcs described with the vertex as a centre, and passing through the points of division. As to the meridians, they are drawn as straight lines through the vertex, and rough points of division equally distant from one another upon the arc f the middle parallel.

This construction is so simple, that it is generally preferred to any er, and the greater part of maps of kingdoms and states are drawn upon this system.

For greater precision, the cone, instead of being taken tangent to the phere, is partially inscribed in it by making it pass through the two xtreme circles of latitude, so that these circles shall be sections of the cone perpendicular to its axis. Imagine a meridian section of the cone and sphere, the angle α formed by the element of this section with the will be measured by half the difference of the arcs included between its (Geom. Ex. 30, p. 48). Supposing a and a' to be the points in which element intersects the meridian section, and λ and λ' their latitudes, $\textsc{n}$ being the place of the north pole, and $\textsc{s}$ the south, the expression for the measure of the above angle will be

$$\alpha = \tfrac{1}{2}\,(\textsc{s}a - \textsc{n}a')$$

$$\text{But } \textsc{s}a = 90^\circ + \lambda, \text{ and } \textsc{n}a' = 90^\circ - \lambda' \quad \therefore \quad \alpha = \tfrac{1}{2}\,(\lambda + \lambda')$$

Now in the right angled triangle formed by the element of the cone, the axis and the radius of the parallel, which last is equal to cos λ, we have for the length of the element terminating at a

$$s = \frac{\cos\lambda}{\sin\frac{1}{2}(\lambda + \lambda')}$$

and for the length of the element terminating at a',

$$s = \frac{\cos\lambda'}{\sin\frac{1}{2}(\lambda + \lambda')}$$

The lengths of the elements of the developed sector being thus known, the rest is as above.

Still better, the cone may be made to pass through two parallels, situated at half distance between the middle parallel and the extremes; the cone would then be partly internal and partly external to the sphere.*

* It was in this way that Delisle constructed the great map of Russia.

PROJECTION OF FLAMSTEED.

This consists in drawing a straight line vertically to represent the central meridian of the map, laying off upon it equal distances say 1°, and through the points of division drawing perpendiculars to this meridian line, which represent parallels of latitude; then laying off upon these parallels distances bearing the same proportion to the distances on the meridian as the cosine of each latitude does to radius unity; finally, drawing through the points of the same graduation, thus determined, curved lines which will represent the other meridians.

The oblateness of the earth may be taken into the account in this method, by laying off in the central meridian not equal distances, but increasing towards the poles in the same proportion as the degrees of the meridian increase. For the demonstration of the formula see App. VI., p. 367. The formula itself is

$$\delta = \frac{\pi a}{180^\circ}\left(1 - \tfrac{1}{4}\,\omega - \tfrac{3}{4}\,\omega \cos 2\lambda\right)^*$$

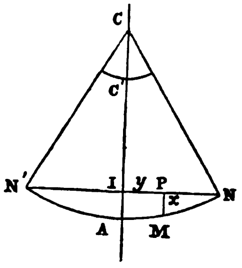
In which

$a = 6377397{\cdot}15$ metres, $\log = 6{\cdot}8046434637$; $\log \omega = 7{\cdot}5233789824$;
$\lambda =$ latitude.

The objection to the method of Flamsteed is that it distorts somewhat the regions distant from the central meridian.

METHOD OF THE FRENCH DÉPÔT DE LA GUERRE.

This is a modification of the conic projection already given, and is that now in use on the coast survey of the United States. The radii of the arcs of circles representing the parallels upon the map being too long to be conveniently described from a centre, they are determined by points. Let there be drawn in the middle of the sheet the perpendiculars CA, NN′; NAN′ represents the middle parallel of the map. Then is known the radius $r =$ CA $=$ R $\cot \lambda$, R being the radius of the earth. Suppose that the map is to



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[*] For a table which gives the length of a meridional arc in any latitude in yards, and a table which gives the length of a parallel, see Lee's Tables and Formulæ, Part II., p. 84.

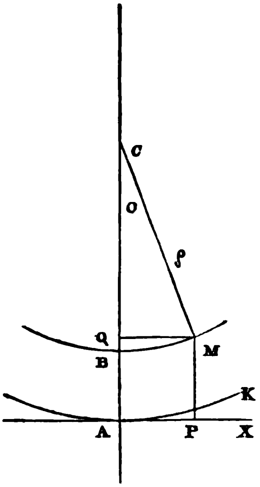
ıbrace D degrees of longitude, the angle C is then known $=$ D $\sin \lambda$. Representing half the chord NN′ by α, CI by β, we have in the triangle CNI

$$\alpha = r \sin \tfrac{1}{2} \text{C}, \ \beta = r \cos \tfrac{1}{2} \text{C}, \ \text{AI} = r\,(1 - \cos \tfrac{1}{2} \text{C})$$
$$= 2r \sin^2 \tfrac{1}{4} \text{C} \text{ [see (7) p. 100].}$$

The extreme points N and N′ of the arc to be described NN′, are thus determined, and the point A, in which it intersects the meridian. Now for other points, such as M, a distance IP $= y$ is laid off from I, and a perpendicular PM is drawn in length equal to x, the value of x being expressed by the following formula

$$x = \sqrt{(r+y)(r-y)} - \beta$$ [*]

The above method does not take into account the earth's oblateness; the following is the generalization of the theory. Let C be the centre of the projection, AK the middle parallel, the latitude of which represent by l; BM another parallel, whose latitude is λ; M the point in question, whose co-ordinates are AP $= x$, PM $= y$, AX being tangent at A, and perpendicular to the principal meridian CA. We have AB $= s$, the distance in latitude between the two parallels, this length s being known by equation (5) p. 367, App. VI. The radius CA $= r$ is also known, being equal to KM in the diagram on p. 365, which, in the right angled triangle KMN, where MN is the normal N, has for its value $r =$ N $\cot l$. Representing the angle ACM by θ, and CM by ρ, we have



$$\text{QM} = x = \rho \sin \theta, \ \text{CQ} = \rho \cos \theta$$
$$y = \text{PM} = \text{BQ} + s = s + \text{BC} - \text{CQ} = s + \rho - \rho \cos \theta = s + \rho\,(1 - \cos \theta)$$
$$= s + 2\,\rho \sin^2 \tfrac{1}{2}\,\theta \text{ [see (8) Art. 72].}$$

We may eliminate ρ from this last by means of the first $x = \rho \sin \theta$. It becomes

$$y = s + x\,\frac{1 - \cos \theta}{\sin \theta} = s + x \tan \tfrac{1}{2}\,\theta \text{ [see App. I., (74) p. 103.]}$$

---

[*] The demonstration of this formula, which requires a knowledge of Analytical Geometry, is as follows:—The equation of the circle, the origin of co-ordinates g at C, is $x^2 + y^2 = r^2$. Transferring the origin to I, the formula for transforma- will be $x = x + \beta$, and the equation of the circle becomes $(x + \beta)^2 = r^2 - y^2$

$$\therefore x = \sqrt{(r+y)\ (r-y)} - \beta$$

The formula in the text.

Dividing therefore N into equal parts, and for each point of division finding the corresponding value of x from the above formula, so many points in the arc NN′ will be determined.

ρ is known since $\rho = r - s$. It remains only to find θ in order to have for each point like M the values of the co-ordinates x and y, viz:

$$x = \rho \sin \theta, \; y = s + x \tan \tfrac{1}{2} \theta$$

N. B.—That s must be taken negative when $\lambda < l$.

The longitude of M estimated from the central meridian suppose to be Λ. This will also be the number of degrees in the arc of the parallel.

But N being the normal at the point M, terminating at the polar axis, the radius of this parallel will be (see diagram, p. 365, App. VI.),

$$N \cos \lambda$$

Moreover, the arc BM of the projection is of the same length with the arc of the parallel, but the number of degrees in two arcs of the same length will be in the inverse ratio of their radii

$$\therefore \Lambda : \theta :: \rho : N \cos \lambda$$

$$\theta = \Lambda \frac{N \cos \lambda}{\rho}$$

which formula serves to determine θ in the same denominations that Λ is given. N is known in terms of λ from formula

$$N = \frac{a}{(1 - e^2 \sin^2 \lambda)^{\frac{1}{2}}}$$

(see App. VI., p. 368), in which $e = 0{\cdot}0816967$, $\log. e = 8{\cdot}9122052271$.

It is easy to perceive now how a map would be drawn according to the projection under consideration. Two lines AC and AX are first drawn at right angles to each other, intersecting at the middle of the sheet A. Setting out from A, we lay off above and below distances such as AB, respectively equal to the values of s, that is to say to arcs of the meridian corresponding to 1°, 2°, 3°, . . . of distance from A, arcs which go on increasing towards the pole.[1] Next we compute the values of the normals N, N′, . . . from degree to degree, the radii ρ of the projected parallels, and finally the amplitudes of the angles θ, which correspond to values of Λ and λ, varying also by degrees, whence result the co-ordinates x and y of the vertices of quadrilaterals in which meridians and parallels of latitude distant from each other, respectively the space of 1° intersect. It remains only to lay off these co-ordinates by a scale of equal parts. The sides of the quadrilaterals joining those vertices thus determined may be drawn without sensible error as straight lines.

The territory to be represented by the map is ordinarily too extended to be placed

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[1] For their values see p. 367, App. VI., also Lee's Tables and Formulas, p. 84, Part II.

I E BY ASTRONOMIC OBSERVATIONS.

OBSERVATIONS FOR LATITUDE WITH ZENITH SECTOR.

The zenith sector employed on the coast survey of the United States determining latitude astronomically, is the same as the mural circle ready described, p. 306, except that only two small portions of the , the one above, the other below the centre, are retained, the rest conceived to be cut away, to render the instrument more portable. mb and telescope, instead of being sustained by a wall, are attached a vertical flat beam of iron, which is capable of reversal about a verti- l axis, and also end for end. Long spirit levels can be attached to the

pon a single sheet. It is customary to form the map by the union, border to border, f a series of sheets, the dimensions of which are 8 decimetres by 5 To find the ns of the vertices of the quadrilaterals upon these sheets, the origin of inates is transferred to one of the corners of the sheet, an operation consists simply in adding or subtracting 1, 2, 3, . . . times 8 decimetres in e direction of the x^s, and as many times 5 decimetres in the direction of the y^s, cording to the place which the sheet ought to occupy in the assemblage. The rder of the sheets is marked upon them. Thus the sheet $\overset{2}{\square}{}_3$ is the one which is second in the horizontal direction, and third in the vertical, estimating from A the intersection of the middle meridian and parallel.

As to the inverse problem, to find the latitude and longitude of a point given upon the map, it will be sufficient to draw through the point lines parallel to the sides of the quadrilateral within which it falls, and to determine upon the scale of equal parts, the values of the fractions which the lines represent.

The following formulas are used on the United States Coast Survey, when the extent of the map is not more than 4° of latitude and longitude.

$$\delta p = (n')\ p \sin \tfrac{1}{2}\,z$$
$$\delta m = (n')\ p \cos \tfrac{1}{2}\,z$$

In which δp is the ordinate $y = \text{BQ}$.

" δm " abscissa $x = \text{QM}$.

" $(n')\,p$ " length of the parallel.

" $\sin \tfrac{1}{2}\,z = \dfrac{(n')\,p}{2\text{T}}$

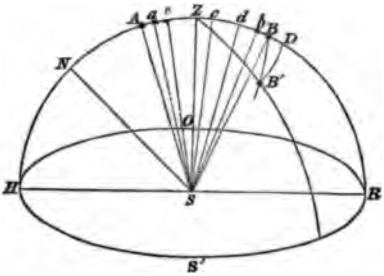
" $\text{T} = \text{N} \cot \text{L}$.

" $\text{N} = \dfrac{a}{(1 - e^2 \sin^2 \text{L})^{\frac{1}{2}}}$

" L = latitude differentiated from.

Here the cone, instead of being assumed tangent to one of the parallels of the map, is supposed successively tangent to each, that it may be required to draw upon back of this bar, in addition to the ghost apparatus in front*. Zenith telescopes are also employed, of similar but less elaborate construction. The practice on the coast survey with these instruments is to observe two stars near the zenith, one north of it, the other south, and differing so little in zenith distance that the difference may be measured with a micrometer.†

The following is a very full exposition of the theory of this method.



In the above figure let

S represent the station occupied.
Z " " Zenith.
N " " Pole.
HORS' " " Horizon.

it. The map thus becomes the developement of the surfaces of several successive cones.

To make the projection, a central meridian is drawn upon the map, along which the lengths of the required minutes are laid off; perpendicular lines are drawn at each of these points, and the values of δp and δm are laid off successively, along and from each of these lines.

Lee's Tables give the values of δp and δm for parallels 30′ apart. The manuscript tables in use on the Coast Survey are computed to every minute. In the diagram, MB, which is very small, is regarded as a straight line, and the angle QMB = C is ½ Z; CM is equal to MK = T in the diagram on p. 365, App. VI., N = MN, and L = MNV in that diagram. With these explanations the student will readily deduce the above formulas for δp and δm.

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* The mode of observing for latitude is similar to that employed with the mural, except that the readings of three levels, one above the other, at the back of the bar, are taken, and the observation is repeated upon the same star in reversed positions of the instrument. The correction for the state of the levels and the reduction to the meridian are made on the principles indicated at pp. 343, 344.

† For description of micrometer see p. 362, App. VI.

In the following investigation let

FOR STAR NORTH OF ZENITH.	FOR STAR SOUTH OF ZENITH.
Δ^n = North Polar Distance.	Δ^s = North Polar Distance.
z^n = Zenith Distance.	z^s = Zenith Distance.
N^n = Reading of north end of Level Scale.	N^s = North End of Level Reading.
S^n = Reading of south end of Level Scale.	S^s = South End of Level Reading.
M^n = Reduction to Meridian.	M^s = Reduction to Meridian.
r^n = Refraction for Star.	r^s = Refraction for Star.

λ = Latitude.

Now if the observation of a star were not affected by refraction, and it were observed at the moment it passed the meridian, and the instrument at the same time were perfect as to level, then the latitude resulting from the observation of a star *north* of the zenith would be expressed by

$$\lambda = 90^\circ - (\Delta^n + z^n) \qquad (1)$$

and for a star south of the zenith by

$$\lambda = 90^\circ - (\Delta^s - z^s) \qquad (2)$$

But, since every observation on a star is affected more or less by refraction, according to its distance from the zenith, and as the instrument is constantly changing (as indicated by the level) during the observations, it becomes necessary that these quantities should be known, and the observations corrected accordingly.

REFRACTION.

Suppose a star to be observed *north* of the zenith, and its north polar distance to be equal to NA in the diagram. If there were no refraction, the star would be observed at A, and the zenith distance would be AZ. But by the effect of refraction, which elevates an object, the star is seen and observed at *a*, consequently the *observed* zenith distance of the star is equal to *az*, which is *too small*, since NA is the true N. P. D. of the star. The measured Z. D. must therefore be *increased* by A*a*, or the amount of refraction.

Hence we have, as a result for latitude by observing a star north of the zenith,

$$\lambda = 90^\circ - (\Delta^n + z^n) - r^n \qquad (3)$$

Having observed the star north of the zenith, the telescope is turned in azimuth 180°, for the purpose of observing the star south of the zenith the N. P. D. of which is equal to NB.

On account of refraction, the star is observed at *b*, and the measured zenith distance is z*b*, which is evidently *too small*, and must be increased by B*b*, the amount of refraction.

For a star south of the zenith we have

$$\lambda = 90^\circ - (\Delta^s - z^s) + r^s \qquad (4)$$

INCLINATION OF VERTICAL AXIS.

If a star *north* of the zenith be supposed to have been observed at *a*, while the instrument was perfectly level, but by a sudden change in the temperature of the atmosphere or any other cause, the instrument is thrown out of level, and the vertical axis takes the line *sc*; since the instrument was perfectly level when the observation was made, the north end of the level scale must have read the *same* as the south end (supposing the level to be in adjustment).

Now, the level being in the direction due north and south, and at right angles to the vertical axis of the telescope, it is evident that when the vertical axis inclines towards the *south*, the *north* end of the level will become *elevated*, and the *south* end *depressed*; and therefore the north end of the level scale will read *greater* than the south end.

Suppose another star of the same north polar distance as the one just observed, should come into the field of the telescope, it would be seen at *a*, and consequently the distance *ac* must be measured with the micrometer, thereby increasing the true Z. D. of the star by the quantity *ae* = *zc*.

Since this distance *zc* is also *measured* by *half* the difference (in arc) of the readings of the level scale (see p. 155), it follows that the level correction will always equal the difference between the *measured* Z. D., and the *true* Z. D. of the star.

Hence we have for latitude, by observation of a star north of the zenith,

$$\lambda = 90^\circ - (\Delta^n + z^n) - r^n + \left(\frac{N^n - S^n}{2}\right) \qquad (5)$$

The telescope is next turned in azimuth 180°, for the purpose of observing a star south of the zenith. The vertical axis still being in the line *sc*, the telescope will take the line s*b*, but the star will be seen at *d*, therefore the *measured* Z. D. will be *too small* by the quantity *bd* = *zc*. The distance zc being measured (as I have before stated), by *half* the dif-ce of the n h and south readings of the level scale, we have for titude by observing a star south of the zenith,

$$\lambda = 90^\circ - (\Delta^s - z^s) + r^s + \left(\frac{N^s - S^s}{2}\right) \qquad (6)$$

"REDUCTION TO MERIDIAN."

were observed before it reached, or after it had passed the n, the observation made would require a third correction, called "Reduction to Meridian."

In the figure suppose the star seen at B to be observed on the meritian, and after it has passed the meridian, to be again observed at B′, then difference between ZB′ and ZB = BD = "Reduction to Meridian."

The formula for "Reduction to Meridian" is

$$m\left(\frac{\cos L \sin \Delta}{\sin Z}\right) = F$$

hich

= tabular quantity taken from the table of "Reduction to Meridian," Table XXXVI. ÷ by sin 1″, the argument being the hour angle or time the star is from the meridian. (See p. 304.) (Lee p. 46)

L = Latitude of Station.

Δ = North Polar Distance of Star.

Z = Zenith Distance of Star.

R = "Reduction to Meridian."

It is evident, without demonstration, that the zenith distance of *any* star will be the *smallest* when it is on the meridian (the telescope being moved in azimuth), therefore the algebraic sign of the correction for "Reduction to Meridian" to the latitude resulting from an observation of a star *north* of the zenith will be +, and for the latitude resulting from the observation of a star *south* of the zenith, the algebraic sign of the correction will be —. Hence for a star *north* of zenith

$$\lambda = 90^\circ - (\Delta^n + z^n) - r^n + \left(\frac{N^n - S^n}{2}\right) + M^n \qquad (7)$$

and for a star *south* of the zenith

$$\lambda = 90^\circ - (\Delta^s - z^s) + r^s + \left(\frac{N^s - S^s}{2}\right) - M^s \qquad (8)$$

By adding (7) and (8) we have for twice the latitude,

$$2\lambda = 180^\circ - (\Delta^n + \Delta^s) - (z^n - z^s) - (r^n - r^s) + \left(\frac{N^s - S^s}{2}\right) + \left(\frac{N^n - S^n}{2}\right) + M^n - M^s$$

or,

$$\text{L} = \frac{180^\circ - (\Delta^n + \Delta^s) - (z^n - z^s)}{2}\, a + \frac{(\text{N}^n + \text{N}^s) - (\text{S}^n + \text{S}^s)}{4}\, b + \frac{(\text{M}^n - \text{M}^s) + (r^s - r^n)}{2}$$

in which a represents the arc value of one division of micrometer, and b the arc value of one division of the level scale.

EXAMPLE.

SURVEY OF THE COAST

I. Latitude, Station Mt. Independence, Cumberland Co., Me.

Date.	No. B. A. C.	N. or S.	Micrometer.		Diff. Z. D.		Polar distances	Twice Approx. Lat'de.	Levels.	
			Turns.	Divisi's.	By Microm.	In arc.			N.	S.
1849.						′ ″	° ′ ″			
Oct. 13	819	N.	3	32·0			37 07 13·32		60·0	43·2
	877	S.	19	62·8			55 33 51·71	87°	44·0	59·5
								′ ″		
					+16 30·8	+12 14·18	92 41 05·23	30 68·95	104·0	102·7
							87 18 54·77			
14	819	N.	3	37·0			37 07 13·27		55·0	49·5
	877	S.	19	66·6			55 33 51·55		49·5	55·0
					+16 29·6	+12 13·64	92 41 04·82	68·82	104·5	104·5
							87 18 55·18			
" 15	819	N.	4	06·0			37 07 13·01		51·0	48·5
	877	S.	20	34·6			55 33 51·39		46·5	53·5
					+16 28·6	+12 13·20	92 41 04·40	68·80	97·5	102·0
							87 18 55·60			
" 18	819	N.	3	10·4			37 07 12·24		42·0	55·2
	877	S.	19	41·2			55 33 50·90		55·0	42·5
					+16 30·8	+12 14·19	92 41 03·14	71·05	97·0	97·7
							87 18 56·86			
" 19	819	N.	2	06·2			37 07 11·99		55·0	47·5
	877	S.	18	32·6			55 33 50·74		49·0	53·6
					+16 26·4	+12 12·20	92 41 02·73	69·47	104·0	101·1
							87 18 57·27			
" 20	819	N.	3	15·0			37 07 11·73		55·0	50·0
	877	S.	19	40·2			55 33 50·58		51·3	53·8
					+16 25·2	+12 11·66	92 41 02·31	69·35	106·3	103·8
							87 18 57·69			

OF THE UNITED STATES.

Observations with Zenith Telescope, No. 3, U. S. C. S.

State of Level.	Meridian distance.	Corrections for twice the Latitude.			Twice Lat'de.	Latitude	REMARKS.
N.—S.		Level.	Merid'n.	Refract.			
	m. s.	"	"	"	87° ′ ″	° ′ ″	
+1·3		+0·58		+0·21	30 69·74	43 45 34·87	
±0·0		±0·00		+0·21	69·03	34·51	Approx. Latitude= ° ′ ″ 43 45 30 cos =9·85870 N.P.D. 37 07 14 sin =9·78067 Z.D. 9 07 16 ar. co. sin=0·79991 x=2·75=0·43928
−4·5	0 24	−2·02	+0·86	+0·21	67·85	33·92	″ ″ ″ m=0·31×2·75=0·86.
−0·7		−0·32		+0·21	70·94	35·47	
+2·9		+1·30		+0·21	70·98	35·49	
+2·5		+1·12		+0·21	70·68	35·34 Mean=34·93	

In the preceding example the 1st column contains the date of the observation, the 2d column the numbers of the stars observed in the British Association Catalogue, the 3d column indicates whether the star is north or south of the zenith; the 4th and 5th columns give the number of turns and fractions of a turn of the micrometer screw, necessary to bring the wire from zero to coincide with the star; the 6th column gives the difference of the micrometer reading, the 7th the value of the same in arc depending, of course, on the value of one turn of the micrometer screw;[*] the 8th column the polar distances of two stars observed on each day, their sum $\Delta^n + \Delta^s$, and $180° - (\Delta^n + \Delta^s)$, according the formula, from which subtracting[†] the number in the 7th column, which is the difference of their zenith distances in arc, or $z^n - z^s$, according to the formula, p. 345, the remainder will be by the formula equal to twice the approximate latitude which is written in the 9th column. The 10th and 11th columns contain the level readings at the north and south ends of the scale, in both positions of the instrument, together with the values of $N^n + N^s$, and $S^n + S^s$, the 12th column shows the difference of these results, or the value of $(N^n + N^s) - (S^n + S^s)$, according to the formula; the 13th column contains the hour angle of the star when not observed exactly on the meridian in minutes and seconds of time; the 14th column contains the correction for error of level, which is obtained by dividing the result in the 12th by 4, and multiplying by the value of one division of the level scale, according to the formula; the 15th column contains the result obtained by the computation in the 19th column, of the correction for the star's not being observed exactly upon the meridian by a method similar to that at top of p. 304, using a more accurate table, in which the constant sin 1″ is incorporated; the 16th column contains the correction for refraction; the 17th column, the double latitude after the corrections in the 16th and 14th have been applied to the 9th; the 18th column, the latitude which is half the result contained in the 17th; the 19th column is for miscellaneous purposes, used in this example for the computation already mentioned.

LONGITUDE BY CELESTIAL OBSERVATIONS.

The best mode of determining longitude ordinarily is by means of moon culminations. The Nautical Almanac gives p. 504 et seq. in the edition of 1850, the apparent R. A. of the bright limb of the moon at the instant of its transit at Greenwich, both for the upper culmination marked U

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[*] A good way of finding the value of one division of the screw head of the micrometer is to note the time by chronometer of the transit of Polaris over the movable wire placed vertically, and set successively to every division of the screw head. Representing by x the angular distance from the meridian at which any reading was taken, by p the hour angle, and by Δ the polar distance of the star, we have

$$\sin x = \sin \Delta \sin p$$

The value of x being computed for each reading, the difference of these values, divided by the difference of the corresponding micrometer readings, gives the value of one division.

[†] Really adding in this example, because z^n is less than z^s.

in the almanac, and for the lower marked L. If now the siderial time of transit be observed at any other station, this will be the ☽'s R. A. at the instant of observation (see p. 151), and the difference will be her variation in R. A. during the interval between the two transits, viz. that over the meridian of Greenwich and that over the meridian of the station. The meridian of the station (supposing it to be W. of Greenwich), has in this interval of time revolved by the diurnal rotation through an angular space equal to the longitude of the station from Greenwich, plus the distance which the moon has moved in R. A. towards the east. To know this angular space, we have only to compute the time occupied by the moon in changing her right ascension by the difference above mentioned. This may be done by means of the variation of the moon's R. A. for one hour, given at the same page of the Almanac* by proportion.

EXAMPLE.

Oct. 8th, 1840, sid. time transit ☽'s I. limb,	23^{h}	1^{m}	$9^{s}\cdot 2$
Error of clock, too slow,			$7\cdot 75$
True time of transit,	23	1	$16\cdot 95$
R. A. ☽'s I. limb (Nautical Almanac, ☽ culm. stars),	23	1	$23\cdot 31$
Difference = var. in R. A.,			$6\cdot 36$

Nautical Almanac gives ☽'s var. in R. A. in 1^{h}, $122\cdot 59$,

$$\therefore\quad 122^{s}\cdot 59 : 1^{h} :: 6^{s}\cdot 36 : 3^{m}\ 6^{s}\cdot 8$$

which last term is the time occupied by the meridian of the station in revolving to that of Greenwich and $6^{s}\cdot 36$ further, the last being the angular motion of the ☽ in R. A., since it made its transit at Greenwich,

$$\therefore\quad 3^{m}\ 6^{s}\cdot 8 - 6^{s}\cdot 36 = 3^{m}\ 0^{s}\cdot 44$$

is the longitude of the place of observation.

The above method requires an exact knowledge of the siderial time. To obviate this necessity, the Almanac also gives the right ascensions of some stars which make their transit nearly at the same time with the moon, and differ little in declination from her, so as to be conveniently observed in connexion th the moon. If the moon had no motion the difference of her right ascension from that of the star would be constant at all meridians; but in the interval of her transit over two different meridians, her right ascension will have varied, and the difference between the two compared differences will exhibit the amount of this variation,* which added to the difference of the meridians shows the angle through which the westerly meridian must revolve before it comes up with the moon. This angle, as before, will be the time in which the moon is undergoing the observed variation in R. A., which may be computed by means of her hourly variation in R. A. given in the Almanac. The variation of R. A. being subtracted from this result, the remainder will be the difference of longitude required.

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* If the distance of the station in long. from Greenwich be great, the variation in R. A. corresponding to the middle interval between the two transits should be used, which may be obtained by interpolation. The numbers in this column of the Almanac include the change of the semidiameter of the limb.

EXAMPLE.

Oct. 8th, 1840, were observed the transits,

Of the ☽'s I. limb,	$23^h\ 1^m\ 9^s{\cdot}2$	R.A. ☽'s I. limb (N. Al.)	$23^h\ 1^m\ 23^s{\cdot}31$
Of the star κ Piscium,	23 18 27·6	R.A. κ Piscium (do.)	23 18 47·98
Difference,	17 18·4	Difference,	17 24·67
			17 18·4

The diff. of the two diffs. = ☽'s var. in R. A. = 6·27

By proportion as before,

$$122^s{\cdot}29 : 1^h :: 6^s{\cdot}27 : 3^m\ 4^s$$

∴ Long. required $= 3^m\ 4^s - 6^s{\cdot}27 = 2^m\ 57^s{\cdot}73$.

When the meridian to be determined is distant from Greenwich, a very simple and unexceptionable way of proceeding is to assume the longitude which is supposed to be known approximately, and from the culminations of the moon's limb, as given in the Nautical Almanac, to find by interpolation the time of culmination of the limb at the assumed meridian. The difference between this and the observed time of culmination will be the interval of time occupied by the moon's limb in passing from the assumed meridian to the true meridian of the station. The motion of the limb in right ascension during this interval must be computed by first determining the hourly motion in right ascension, by interpolation, for the instant of passing the assumed meridian, and proceeding by proportion, as in the examples above. The result thus obtained being subtracted from the interval, the remainder will be the difference of longitude between the assumed meridian and the meridian of the station.

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* For the determination of this variation with great accuracy, observations should be taken simultaneously at the different meridians to be compared. Errors in the computed places of the moon or stars are thereby avoided. The results given in the Almanac may be considered as a very near approximation to what would have been the indication of the Greenwich instruments, had the observations actually been made with them. The traveller has thus the opportunity of rendering his observation immediately available for determining his longitude with considerable accuracy.

The formula for interpolation is

$$y = \text{A} + \frac{m}{n}\,\delta + \tfrac{1}{2}\,\frac{m}{n}\left(\frac{m}{n} - 1\right)\delta_2 + \tfrac{1}{6}\,\frac{m}{n}\left(\frac{m^2}{n^2} - \frac{3m}{n} + 2\right)\delta_3 + \&c.$$

In which A is the element for the noon, midnight, or complete hour preceding the given instant,

y is the element required for the given time,

m the given number of hours since noon or midnight, or minutes since the even hour (the long. in time of the assumed meridian above).

n is 12 hours, 24 hours, or 60 minutes, the interval between the times, for which the element is given in the Nautical Alm.

δ the difference between two consecutive elements in the Naut. Alm.

δ_2 the difference between the successive values of δ,

δ_3 " " " " " " " δ_2, &c.

For a convenient mode of proceeding, and an example under it where the meridian is distant from Greenwich, see Lee's Tables and Formulas, pp. 69—78, Part III.

LONGITUDE BY ECLIPSES OF JUPITER'S SATELLITES.

The eclipses of Jupiter's Satellites, especially the first, afford the readiest mode of obtaining the longitude, both from the frequent occurrence of the phenomena, and the simplicity of the calculation.

All that is necessary to be known is the exact time of observation; the difference between this time and the time at Greenwich* shows the difference of longitude, and is *east* or *west* of Greenwich, according as the time of observation is *greater* or *less* than the Greenwich time.

EXAMPLE.

Suppose the emersion of Jupiter's first satellite to be observed August 8th, 1850, at Paris, and the time of observation there to be 14^h 30^m

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* This is given at p. XX. of the Nautical Almanac for each month. At p. 605 of the edition of 1850 is a full description of the page and its use.

me. e. The emersion takes place at Greenwich (Naut. Alm., p. XX.), at $14^h\,20^m\,55^s{\cdot}8$ Greenwich mean time; the difference $9^m\,21^s{\cdot}5$ is the difference of longitude between Greenwich and Paris. And because the time at Paris is greater than that at Greenwich, the former is east of the latter.

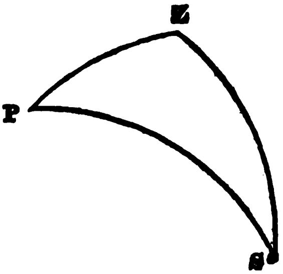
ASTRONOMICAL DETERMINATION OF AZIMUTHS.

In the previous pages the methods of determining difference of azimuths geodetically, or from the triangulation, have been given. But the usefulness of these methods depends on the implied ability to obtain by astronomic observation the azimuths of certain lines from which the others are differentiated.

The method of proceeding is to determine by observation the difference of azimuth between the sun or a star, and the line whose azimuth is to be determined, then to find by calculation the azimuth of the sun or star; the sum or difference of these results will be the azimuth required. The difference of azimuth between the sun or star and the line whose azimuth is to be observed is obtained with an altitude and azimuth instrument, or theodolite. The middle vertical wire is made to bisect the star, or to touch the limb of the sun, and the siderial time is observed at the same instant; the reading is then taken on the horizontal limb of the instrument, which is afterwards turned to a signal (bearing a lamp, if at night), which is placed upon one of the sides of the triangulation, or upon any other convenient line, the horizontal angle between which, and the line whose azimuth is required, can be subsequently measured, and the reading of the horizontal limb again taken. The difference of the two readings will be the difference of azimuth between the sun's limb, or star and the signal, at the instant of siderial time above mentioned. With the altitude and azimuth instrument, the transit of both limbs of the sun, or the transit of the star, may be taken over all the wires of the instrument, and the mean of the times taken as the time at which the azimuthal position of the sun's centre, or the star, corresponded to the reading of the horizontal limb.

AZIMUTH OF THE SUN OR A STAR.

The determination of this requires merely the solution of the triangle ZPS, in which PZ the colatitude of the place of observation, PS the polar distance of the sun or star, and P the hour angle equal to the difference



between the right ascension of the object and the siderial time of observation are given to compute the angle z, which is the azimuth required. Where the object is the sun, of course the value of the semidiameter at the instant must be computed and applied to the reading for the sun's limb to obtain that for his centre. If the altitude of the object is also observed at the same instant, or immediately before or after, and reducéd to the instant by interpolation, one of the above data, either the hour angle or the latitude, may be replaced by the zenith, distance zs in the triangle.

The formulæ will be

$$\tan \tfrac{1}{2}\,(z + s) = \cot \tfrac{1}{2}\,P\,\frac{\cos \tfrac{1}{2}\,(\varpi - \lambda)}{\cos \tfrac{1}{2}\,(\varpi + \lambda)}$$

$$\tan \tfrac{1}{2}\,(z - s) = \cot \tfrac{1}{2}\,P\,\frac{\sin \tfrac{1}{2}\,(\varpi - \lambda)}{\sin \tfrac{1}{2}\,(\varpi + \lambda)}$$

$$z = \tfrac{1}{2}\,(z + s) + \tfrac{1}{2}\,(z - s)$$

In which

z = azimuth required.
P = hour angle.
ϖ = polar distance of object.
λ = colatitude of station.

The formula, if the siderial time be unknown, and the altitude observed is

$$\cos \tfrac{1}{2}\,z = \sqrt{\frac{\sin \tfrac{1}{2}\,s \sin (\tfrac{1}{2}\,s - \varpi)}{\sin \zeta \sin \lambda}}$$

$$s = \zeta + \varpi + \lambda$$

In which

ζ = observed zenith distance.

AZIMUTH BY POLARIS.

The best mode of obtaining the azimuth of a line upon the surface of the earth is by means of the pole star when at its greatest eastern or western elongation. With a telescope as powerful as that of the great theodolite, the necessary observations may be conducted in the day time, the star being distinctly visible. The mode of proceeding is to commence about 15 minutes before the time of greatest elongation, and place the middle vertical wire alternately upon the star, and upon a signal nearly in the direction of the meridian, a mile or two distant, illuminated if the observation be at night. The readings are taken by the micrometer microscope, on the horizontal limb, both when the middle wire is upon the star and upon the signal, the difference of azimuth of which will be indicated by the difference of the reading, so that when the azimuth of Polaris, at the instant of each observation upon it, is known, the azimuth of the signal becomes known; the mean of all the results is taken as the true azimuth, and thus a line whose azimuth is fixed becomes determined on the ground, from which other azimuths may be differentiated.

The following is the mode of determining, at any instant, the

AZIMUTH OF POLARIS.

If we suppose a spherical triangle having for its three vertices the zenith, the pole, and the star; this triangle, at the time of the star's greatest elongation, will be right angled at the star; for if a cone be conceived having its vertex at the eye of the observer, and for its base the diurnal circle of the star, the tangent plane to this cone, passing through the star, is perpendicular to the declination circle through the star, which is a meridian plane of the cone; the visual or tangent plane through the star at its greatest elongation being a vertical plane, passes through the zenith, and, also passing through the star, determines on the celestial sphere a side ZS of the spherical triangle ZSP, so that the angle at S is therefore a right angle. In this right angled triangle are known ZP the colatitude of the station, and PS the polar distance of the star, to find the hour angle P, and the azimuth Z, at the time of greatest elongation. The former, applied to the time of the star's meridian transit or R. A. will give the time of greatest elongation. The formulas are

$$\text{For the hour angle} \cos P = \tan \varpi \cot \lambda$$
$$\text{For the azimuth,} \sin Z = \sin \varpi \operatorname{cosec} \lambda$$

In which ϖ = polar distance, λ = colatitude.

If the star be observed within 45^m of the time of the greatest elongation, the observation may be reduced by the formula

$$c = \frac{225}{2} t^2 \sin 1'' \tan z$$

in which c is the correction of the azimuth, t the siderial time from elongation, and z the greatest azimuth. The correction is deduced in a manner similar to that on p. 302. Table XXXVI. may be made available as explained at the bottom of that page, or the constant log. 112·5 sin 1″ = 6·7367274 may be used with the logs. of t^2 and tan z. This correction being applied subtractively to the azimuth at the time of greatest elongation, computed as above, will give the azimuth at the time of observation.

If the axis of the telescope be not horizontal, the correction for azimuth is, *d* being the value of one division of the level scale,

$$d\,\frac{(\omega + \omega') \sim (e + e')}{4}\ \tan \ast\text{'s alt.}$$

TRIGONOMETRIC LEVELLING.

This consists in observing the zenith distances of two stations, and applying the corrections for curvature and refractions, to obtain their difference of level. The theory is simple, and the necessary formulas and table are found at pp. 50 to 54, Part I. of Lee Tables and Formulas.*

The usual mode of observing zenith distances is as follows: the instrument is carefully levelled, *i. e.*, the vertical axis is placed truly vertical; the horizontal wire of the telescope is then pointed at the object, and the vertical circle read off; next the instrument is revolved 180° in azimuth, and the telescope being then moved through the double zenith distance of the object, is pointed again. If we now read off, the difference between the two readings will be 2 Z. D.; the operation is, however, repeated (generally six times) if the vertical limb has the repeating motion, before it is read off again. The instrument should be levelled for each set of observations.

MAGNETIC OBSERVATIONS.

These usually accompany the operations of a Geodetic survey. They have for their object to determine, 1. The angle which the magnetic meridian makes with the astronomic meridian, commonly called the variation of the needle, but more properly the *Declination*. 2. The angle under which a needle suspended by a perfectly flexible thread at its centre of gravity, would be inclined to the horizon, commonly called the dip, but more properly the *inclination;* and 3. The *intensity* of the magnetic force at any place; with the daily and other periodical variations in these three elements.

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* Immediately following (p. 55) are formulæ and tables for the barometric measurement of heights.

The instrument for observing the declination is called a *declinometer* or *declination magnetometer.* Where only the variation of the declination is to be observed, the instrument consists of a horizontal telescope, firmly supported, pointing towards a magnetic needle bearing a mirror so adjusted as to reflect a horizontal scale placed directly under the object glass of the telescope. The least change in the direction of the needle will be indicated by a change in the reading of the scale marked by the middle vertical wire of the telescope.

The best form of instrument for the measurement of absolute declination is a theodolite, or altitude and azimuth instrument, in front of which is suspended a collimator magnet, by fibres of untwisted silk, resting horizontally in a stirrup of gun metal. The collimator magnet is a hollow cylindrical magnet, with a small object glass like a telescope, and a horizontal scale at its focus.

The adjustments of this instrument consist in bringing the collimator magnet into the magnetic meridian without torsion of the thread; in determining the zero division of the scale corresponding to the magnetic axis of the collimator magnet; and in bringing the line of collimation of the theodolite telescope into the magnetic meridian, its vertical wire coinciding with this division. These adjustments are all made at once, by putting in a bar first, equal in weight to the collimating magnet, and adjusting the stirrup approximately; then, after restoring the collimating magnet by repeated trials, making half the necessary correction by moving the theodolite in azimuth, and half by turning the torsion screw at the top of the thread, till the same division of the scale is read, with the collimator in two positions, the second position being produced by turning the collimator over, so that it shall have revolved 180° about its optical axis. There is then no torsion of the thread, the axis of the collimator magnet and of the theodolite are both in the magnetic meridian, and the division read is the zero of the scale.

If in this position the verniers of the azimuth circle of the theodolite be read, and if its telescope then be turned in the direction of some object, whose azimuth is known or can be afterwards determined, the difference of the reading, added to or subtracted from the azimuth of the object, will give the absolute declination. The angular value of one division of the scale is determined by measuring with the theodolite the horizontal angle subtended by a certain number of the divisions, the magnet being temporarily fixed.

If a denote the angular value of one division of the scale, and $\frac{\mathrm{H}}{\mathrm{F}}$ the ratio of the torsion and magnetic forces, the true declination changes are deduced by multiplying the observed differences of reading by $a\ (1 + \frac{\mathrm{H}}{\mathrm{F}})$. The value of $\frac{\mathrm{H}}{\mathrm{F}}$ is determined by turning the torsion through two large angles, and noting the corresponding differences of reading. If ω denotes the angular value of the former, and u that of the latter,

$$\frac{\mathrm{H}}{\mathrm{F}} = \frac{\omega}{\omega - u}$$

ABSOLUTE HORIZONTAL INTENSITY

Requires for its determination, 1, experiments of *deflexion*, 2, of *vibration*. The former give the *ratio* of the magnetic moment of the deflecting magnet to the horizontal intensity, the latter the *product* of the same quantities, and their separate value is obtained by algebraic elimination.

Experiments of deflection consist in placing a magnetic bar, called a deflector, at one side of a freely suspended magnet, in a line drawn horizontally through the centre of the suspended magnet, perpendicularly to the magnetic meridian, its axis coinciding with this line. The deflector should be placed at three different distances from the suspended magnet on this line, in direct and reversed positions, or turned end for end, at each.

Experiments of vibration consist in suspending the same magnet which was used as a deflector, and noting the times at which some central division of the scale passes across the vertical wire of the telescope, at the beginning and end of at least 300 horizontal vibrations of the magnet, the magnet vibrating steadily in a very small arc.

As the time of vibration depends on the form and weight of the suspended mass as well as upon the product of the magnetic moment and horizontal intensity, its moment of inertia must be ascertained by means of a series of vibrations with two cylindrical weights of equal dimensions, whose moment of inertia is known, at opposite ends of the magnet.

If m denote the magnetic moment of the deflecting magnet, X the horizontal intensity, the formulas are

$$\frac{m}{\mathrm{X}} = \tfrac{1}{2}\, r^3 \tan u$$

$$mx = \frac{\pi^2 K}{T^2}$$

he of which

between centres of deflecting and suspended magnets in feet

ls.

lection obtained by multiplying half the mean of each

r ult by the coefficient (see above), $a\ (1 + \frac{H}{F})$

In the second

$\pi = 3{\cdot}1416$

$K = \frac{T^2}{T'^2 - T^2} K'$

T' = time of vibration with weights.

T = " " without "

nt of inertion of the weights is

$$K' = (\tfrac{1}{2} l^2 + 2r'^2)\, p$$

In which l = interval of the points of suspension.

" r' = radius of the cylinders in decimal of a foot.

" $2p$ = their mass in grains.

THE INCLINATION

is found directly by the dipping needle, which consists of a magnetic needle, suspended at the centre of a graduated vertical circle. The mean must be taken of results with several needles, reversed on their magnetic axes, and reversed as to their poles by remagnetizing. Observations should be made at different azimuths to test the limb of the instrument, which is often magnetic, in which use the formula

$$\tan \theta = \tan \eta \operatorname{cosec} a$$

In which θ = inclination sought.

" η = " observed.

" a = azimuth of the vertical circle.

The inclination may also be found by means of the horizontal and vertical components of the intensity, as it would be determined by the direction of their resultant. The vertical component is observed by means of a vertical force magnetometer, which is a needle suspended like the dipping needle, but placed in a plane perpendicular to the magnetic meridian, and made to vibrate in this plane.

On the other hand the vertical component may be deduced from the inclination and the horizontal component.

For further information on this subject see Lee's Tables and Formulæ, and Riddell's Magnetical Instructions. For the theory of magnetism as applicable, see the papers of Gauss, and a late elementary work of Prof. Lamonte, of Munich. Translations of some of the papers of Gauss, Lamonte, and Weber have been published in Taylor's Scientific Memoirs, Parts 5, 6, 11, and 12.

APPENDIX TO PART VI.

INSTRUMENTS FOR EXTRA MERIDIAN OBSERVATIONS.

THE principal of these are the *equatorial*, and the *altitude and azimuth instrument*.

THE EQUATORIAL

is a telescope usually of large size, upon an equatorial mounting. The latter consists, 1, of a strong metallic axis, placed in a position parallel to the axis of the earth, so as to point to the pole of the heavens, the lower end of this axis, which is called the polar axis, being enlarged into a circle called the hour circle, the plane of which is parallel to the equator, and the circumference of which is divided into hours and fractions of an hour. 2. Of another axis crossing the upper end of the former at right angles, called the equatorial axis, one end of which is enlarged into a circle called the declination circle, divided into degrees and fractions of a degree, and firmly fastened to the other end of which, at right angles, near the middle of its tube, is the telescope. The instrument must be so adjusted that when the optical axis of the telescope describes the meridian as the instrument moves upon the equatorial axis alone, the index of the hour circle is at the zero, and when the optical axis points to a star in the equator, the index of the declination circle is at zero. Then if the instrument be turned on its equatorial axis till the index or vernier of the declination circle points to the declination of any celestial object as given by the Nautical Almanac or by catalogue, and on its polar axis till the index of the hour circle points to the hour angle, which is the difference between the right ascension and the time by the siderial clock, the object will be seen in the centre of the field of view of the telescope. As it passes out of the field of view by the rotation of the earth on which the instrument stands, the telescope is made to follow it by a rotation of the instrument on its polar axis alone. Attached to this axis is a clamp and screw of slow motion for the purpose. Sometimes the polar axis is made to move by clockwork, the velocity being regulated by friction, and the motion becoming uniform when the friction is equal to the accelerating force of the clock-weights. In the Fraunhofer* mounting, a hollow inverted frustum of a cone contains balls, supported at the ends of a flexible bar at right angles to the axis of the frustum, about which it revolves, carrying the balls which rub against the sides of the frustum. The velocity should be different for the sun, moon, each of the planets, and for the fixed stars. If the bar be lowered in the frustum, less velocity will make the friction equal to the accelerating force. If raised, more velocity will be required. For each kind of heavenly body the requisite velocity is determined by experiment, and a permanent mark made where an attached index stands.

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* So called, from the inventor and first manufacturer, Fraunhofer, of Munich.

Approximate adjustment is sufficient for this instrument,[1] which is ordinarily used as a differential instrument, by means of an appendage which we proceed to describe, called

THE POSITION MICROMETER.

This is an eye-piece which screws on in place of the ordinary eye-piece of the telescope, and consists of a circle of brass about four inches in diameter, the plane of which is perpendicular to the optical axis, which passes through its centre. It is graduated on the outer rim, the graduation being numbered to 360°. A rectangular box, about one inch by four, and the eighth of an inch in thickness, is fitted to the circle in the position of a diameter, at the ends of which are micrometer screws, which move each one of two parallel wires along a notched scale, the wires and scale being seen (when the eye is applied to the telescope) at the focus of the object glass, where also the image of the heavenly body is formed. The circle carrying the box has also a motion round the optical axis, by means of a screw projecting perpendicularly to the plane of the circle, which acts as a pinion by cogs, in a cog-wheel of less diameter than the circle, attached to the piece which screws into the telescope to which two verniers, 90° apart, marked A and B, are firmly fixed.

To determine the right ascension and declination of a new heavenly body, as a comet, for instance, with this instrument in any part of the visible heavens, let the object be brought into the field of view at the same time with some fixed star, one whose place is given by catalogue, if possible. Bring the star to one of the movable micrometer wires, and turn the micrometer in position, *i. e.* round the optical axis till the wire threads the star in its motion along its diurnal path. The wire is then parallel to the equator. Let the two wires now be separated by turning the micrometer screws till one of them passes through or bisects the star, and the other bisects the comet; the number of turns of the screw shown by the notched scale, and the fractions of a turn by the screw-head, will indicate the difference in declination between the comet and the star.* Let the micrometer now be turned in position 90°, and the transits of the comet and star across the two wires be observed by the siderial clock. The difference in the times of transit will be the difference of right ascension of the comet and star. The absolute right ascension and declination of the comet thus becomes known, if that of the *star of comparison* be known from catalogue. If this be not the case, the star must be brought to the centre of the field of view indicated by the point at which a third wire at right angles to the other two crosses them when they are made to coincide at the zero of the notched scale, and the approximate right ascension and declination of the star must be noted by the declination and hour circles and clock, with sufficient accuracy to identify it, in order that its place may be more exactly determined by observation with the meridian instruments, the transit and mural, at some subsequent time.

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* It will be found convenient to make the wires coincide before commencing the operation with the screwhead of one of them at the zero, and let this be the only one moved, if possible, and read by its screw.

[1] For methods of exact adjustment, see Lond. Ast. Soc. Memoirs, vol. IV. p. 495.

The micrometer screw head is divided into 100 parts. The value of a single turn of the screw in arc is determined by measuring the diameter of a planet, given by the Nautical Almanac or the known distance apart of a pair of stars, and then by the proportion, as the number of turns and hundredths of a turn of the micrometer screw is to the known distance measured, so is 1 to the value of one turn.

MEASUREMENT OF ANGLES OF POSITION AND DISTANCE OF DOUBLE STARS.

The angle of position of a pair of stars is the angle which the visual plane passing through both the stars makes with the plane of the declination circle passing through the larger star. It is estimated from the s. round by the w. to 360°. The following is Capt. Smythe's method of observing position and distance.

Bring the wires coinciding at the zero of the scale, with the index of one screw-head at the zero, upon the line of the two stars, so as to bisect both, and read one of the verniers; next turn the instrument 90° in position, and measure the distance of the stars apart; finally turn in position till one of the stars runs along either of the wires, and read one of the verniers again, the difference between the first and last vernier reading, will give the angle which the visual plane of the two stars makes with the equator, from which the angle of position may be obtained in an obvious manner. In the transit instrument and instruments of that class the wires are made visible at night by a lamp placed at one end of the supporting axis, which is left open for the purpose, with a piece of glass over it; the light is received by a small plane mirror in the axis of the telescope, and reflected down the tube to the wires. In the equatorial instrument the horizontal tube bearing the lens (colored red) of a small lamp is inserted in the tube of the telescope, near the wires of the micrometer, and a reflector so arranged as to throw the light on the wires.

When the object to be observed is so faint as not to bear illumination, a ring micrometer is used. This is a black circle on a piece of plane glass in the focus of the object glass, with which differences of right ascension and declination are obtained by noting the times occupied by the two objects to be compared in crossing the circle. Half the sum of the times of either object's making the transit of the circumference on opposite sides will be the time of its passing the middle diameter, and the difference of the time of passing the middle diameter by the two objects will be their difference of right ascension.

For declination it is necessary to ascertain, by experiment, the time of an equatorial star's passing over a diameter of the ring, by observing the time of any other star, and multiplying by the cosine of the declination. Then the ratio of the time occupied by a star in passing over a chord of the ring to the time which it would occupy in passing over the diameter is the cosine of an angle, the sine of which to the radius of the ring is the difference of declination between the centre of the ring and the star. (For the whole theory see Ast. Nach., Vol. 8.)

THE ALTITUDE AND AZIMUTH INSTRUMENT.

This is in effect a large theodolite, with two micrometer microscopes, 180° apart, on both the horizontal and vertical limb. It may be used as a theodolite, also as a sit instrument. It may be used like the zenith sector, for determining latitude by a star nearly on the meridian, the instrument having a spirit level parallel to the plane of the vertical circle.

The following is a good form.

No. of observation.	Date.	Object.	Approx. zen. dist.	North or South.	Direct or reflex.	Face E. or W.	Readings.							Error of time keeper.	Barometer.	Thermom.	Remarks.
							Time Keeper.	Level.	Microscopes.								
									A		B		Mean.				
									Reading.	Corr. for run.*	Reading.	Corr. for run.					
			° ′				h. m. s.		° ′ ″	″	″	″	° ′ ″	s			

There are five vertical and five horizontal wires, the latter of which are convenient for observing single altitudes, or equal altitudes of the sun for either time or latitude. The mean of the times of the two limbs passing the five horizontal wires is taken as the time of the altitude shown on the vertical limb.

A very accurate mode of determining the true time and error of a time keeper is by equal altitudes, morning and afternoon. If the sun did not change his declination in the interval between the two observations, half the interval in time added to the time of the morning observation would express the hour by the time keeper, when the sun was on the meridian. But as the declination does change, a correction of the half interval must be made, the formula for which is

$$d\,\tfrac{1}{2}\,p = d\delta\,(\tan\delta\,\cot\tfrac{1}{2}\,p - \tan\lambda\,\operatorname{cosec}\tfrac{1}{2}\,p)$$

In which

$d\,\tfrac{1}{2}\,p =$ the correction of the half interval,
$d\delta =$ the change of declination in the half interval,
$\lambda =$ the latitude (— if south).

The correction is + if the declination is increasing, and — if decreasing.

The time of the sun's being on the meridian being corrected for the equation of time will give the time of mean noon by the watch, which will show the error of the watch.

If a star be used instead of the sun, no correction is requisite for change of declination, the mean between the two times of observation must be compared with the computed mean time at which the star culminates, in order to have the error of the time keeper.

If the readings be taken on the horizontal circle at the two times of observation, the reading midway will correspond to the direction of the meridian. This also, in the case of the sun, requires a correction for change of declination in the interval, the formula for which is

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* This is for error in the divisions of the limb, tested by running the microscope over them.

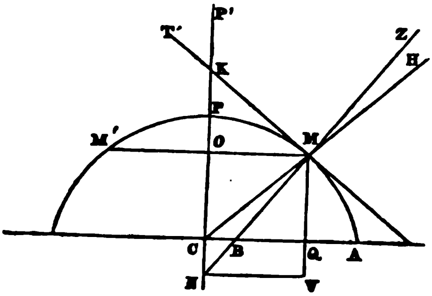
$$dz = \frac{\cos \delta}{\sin z \cos \lambda \cos a}$$

In which

$dz =$ correction of azimuth.
$a =$ observed altitude.

CONVERSION OF ASTRONOMIC AND GEOCENTRIC LATITUDE.

From the nature of the astronomic instruments, the zenith point being determined by a plumb line, basin of mercury, or spirit level, it follows, by the law of gravitation, that the line from the station to the zenith is a normal to the elliptical meridian, and the angle which this line makes with the major axis or equatorial diameter will be the astronomic latitude, or latitude deduced from observation. This will be the angle MBA, in the diagram.



The expression for the subnormal BQ is

$$\frac{B^2 x}{A^2}$$

and the ratio of this to CQ $= x$ is

$$\frac{B^2}{A^2}$$

But from the diagram we have the proportion,

$$CB : BQ :: \tan CMQ : \tan BMQ$$
$$\therefore \quad A^2 : B^2 :: \cot MCQ : \cot MBQ$$

But if the ratio of A to B be taken as 305 to 304, then

$$\frac{B^2}{A^2} = \cdot 9934$$

and the formula for converting astronomic into geocentric latitude will be

$$\tan \lambda' = \cdot 9934 \tan \lambda$$

In which

$\lambda =$ astronomic latitude,
$\lambda' =$ geocentric latitude.

APPENDIX VI.

RADIUS OF CURVATURE IN TERMS OF THE LATITUDE.

We have had occasion to use an expression for the radius of curvature* of the meridian considered as an ellipse, in terms of the latitude of the point of the meridian, under consideration, in various places in the part of this volume devoted to Geodesy. The following is the mode of deriving it.

The ordinary expression for the radius of curvature of the ellipse found in elementary mathematical works is

$$\rho = \frac{(a^4 y^2 + b^4 x^2)^{\frac{3}{2}}}{a^4 b^4} \qquad (1)$$

in which a and b denote the semi-axes of the ellipse x and y, tne co-ordinates of the point at which ρ is the radius of curvature. If λ denote the latitude of this point, since it is the angle which the normal makes with the major axis, we have

$$\tan \lambda = \frac{a^2 y}{b^2 x} \quad \therefore \quad \tan^2 \lambda = \frac{a^4 y^2}{b^4 x^2} \quad \therefore \quad \sec^2 \lambda = \frac{a^4 y^2 + b^4 x^2}{b^4 x^2}$$

whence,

$$\frac{\tan^2 \lambda}{\sec^2 \lambda} = \frac{a^4 y^2}{a^4 y^2 + b^4 x^2} = \sin^2 \lambda$$

But (e denoting the excentricity of the ellipse),

$$e^2 = \frac{a^2 - b^2}{a^2} \; (2) \quad \therefore \quad e^2 \sin^2 \lambda = \frac{a^2 (a^2 - b^2) y^2}{a^4 y^2 + b^4 x^2}$$

$$1 - e^2 \sin^2 \lambda = \frac{b^4 x^2 + a^2 b^2 y^2}{a^4 y^2 + b^4 x^2} = \frac{b^2 (b^2 x^2 + a^2 y^2)}{a^4 y^2 + b^4 x^2} = \frac{a^2 b^4}{a^4 y^2 + b^4 x^2} \qquad (m)$$

since the part of the numerator in parenthesis is, by the equation of the ellipse, equal to $a^2 b^2$. From the last equation we obtain

$$(a^4 y^2 + b^4 x^2)^{\frac{3}{2}} = \frac{a^3 b^6}{(1 - e^2 \sin^2 \lambda)^{\frac{3}{2}}}$$

Substituting the second member of the last result for the numerator of (1) that formula becomes

$$\rho = \frac{b^2}{a} \quad \frac{1}{(1 - e^2 \sin^2 \lambda)^{\frac{3}{2}}} \qquad (3)$$

But from (2) $b^2 = a^2 (1 - e^2)$, therefore (3) becomes

$$\rho = a \frac{1 - e^2}{(1 - e^2 \sin^2 \lambda)^{\frac{3}{2}}} \qquad (4)$$

DETERMINATION OF THE FIGURE AND DIMENSIONS OF THE EARTH FROM THE MEASUREMENT OF TWO DEGREES OF THE MERIDIAN AT TWO DISTANT LATITUDES.

The deviation of an oblate spheroïd from a sphere is expressed by what is termed its compression or oblateness. This is the ratio of the difference between its axis to the major axis in symbols, ω representing the oblateness,

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* The radius of curvature of a curve at any point is the radius of a circle having the same curvature as the curve at that point.

$$\omega = \frac{a - b}{a} \qquad (1)$$

$$\therefore \quad b = (1 - \omega)\, a$$

But (2) p. 366,

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{a^2 - (1 - \omega)^2 a^2}{a^2} = 1 - (1 - \omega)^2 = 2\omega - \omega^2$$

omitting ω^2, which is a very small fraction, in consequence of the smallness of ω, we may write

$$e^2 = 2\omega \qquad (2)$$

But from (4) p. 366, applying either the binominal or McClaurin's theorem to the second member, we have

$$\rho = a\,(1 - e^2 + \tfrac{3}{2}\, e^2 \sin^2 \lambda + \text{terms too small to affect the result}) \qquad (3)$$

By formula (5) p. 100, $\cos 2\lambda = 1 - 2 \sin^2 \lambda$,

$$\therefore \quad \frac{\sin^2 \lambda}{2} = \frac{1 - \cos 2\lambda}{4}$$

substituting the second member of this last in place of the first in (3), that equation becomes

$$\rho = a\,(1 - \tfrac{1}{4}\, e^2 - \tfrac{3}{4}\, e^2 \cos 2\lambda + \&c.)$$

or by (2),

$$\rho = a\,(1 - \tfrac{1}{2}\,\omega - \tfrac{3}{2}\,\omega \cos 2\lambda) \qquad (4)$$

If now δ denote the length of a degree measured in the latitude λ, since ρ is the radius of the arc of the meridian in that latitude,

$$\delta = \frac{2\pi\rho}{360} = \frac{\pi\rho}{180}$$

hence substituting the value of ρ given by (4)

$$\delta = \frac{\pi a}{180}\,(1 - \tfrac{1}{2}\,\omega - \tfrac{3}{2}\,\omega \cos 2\lambda) \qquad (5)$$

If δ' denote the length of a degree measured in another latitude λ', in a similar manner

$$\delta' = \frac{\pi a}{180}\,(1 - \tfrac{1}{2}\,\omega - \tfrac{3}{2}\,\omega \cos 2\lambda')$$

$$\frac{\delta}{\delta'} = \frac{1 - \frac{1}{2}\,\omega\,(1 + 3 \cos 2\lambda)}{1 - \frac{1}{2}\,\omega\,(1 + 3 \cos 2\lambda')}$$

Performing the division in the second member, and neglecting the squares and higher powers of ω we have

$$\frac{\delta}{\delta'} = 1 + \tfrac{3}{2}\,\omega\,(\cos 2\lambda' - \cos 2\lambda)$$

fore

$$\omega = \tfrac{2}{3}\,\frac{1}{\cos 2\lambda' - \cos 2\lambda}\quad\frac{\delta - \delta'}{\delta'} \tag{6}$$

Hence this *rule.*

ction which shall be the ratio of a degree in one latitude to its excess in another latitude.

ion, which shall be the ratio of unity to the difference between of the doubles of each latitude.

Take ⅔ of the product of these two fractions.

The value of ω being known, that of *a* may be found from (5).

EXAMPLE.

f a degree at the equator is $\delta' = 56{\cdot}753$ toises.

" at 45° $\delta = 57{\cdot}008$ "

$$\therefore\ \cos 2\lambda = 0$$

$$\cos 2\lambda' = 1$$

$$\frac{\delta - \delta'}{\delta} = \frac{{\cdot}255}{57{\cdot}008}$$

$$\omega = \frac{2}{3}\cdot\frac{{\cdot}255}{56{\cdot}753} = \frac{{\cdot}510}{170{\cdot}259} = \frac{{\cdot}255}{85{\cdot}129} = {\cdot}00299.$$

We have had occasion to use the expression

$$N = \frac{a}{(1 - e^2 \sin^2 \lambda)^{\frac{1}{2}}}$$

in which N denoted the *y* normal, and λ the latitude.

This formula may be deduced as follows:—The expression for the *y* normal in the ellipse is

$$y \text{ normal} = \left(\frac{a^4 y^2 + b^4 x^2}{b^4}\right)^{\frac{1}{2}}$$

But we have seen (*m*) p. 366, that

$$\frac{a^4 y^2 + b^4 x^2}{a^2 b^4} = \frac{1}{1 - e^2 \sin^2 \lambda}$$

whence the formula required.

PRIME VERTICAL TRANSIT.

A transit instrument, mounted in the prime vertical, or at right angles to the meridian, affords a very accurate method of determining latitude or declination when either is known, by observing the times of transit of the same star over the prime vertical on both sides of the zenith. Stars near the zenith are the best for this purpose, as the interval between the two transits is shortest for them, and there is less opportunity for instrumental changes between the two observations. The triangle PZS, which we have so often used in the preceding pages, which must, for our present purpose, be considered right angled at Z, and in which P = the half interval between the transits, will produce the requisite formulæ in a very simple manner. They are as follows:

$$\cos \text{ of } \tfrac{1}{2} \text{ interval reduced to arc} \times \cot \text{dec.} = \cot \text{lat.}$$

$$\sin \text{alt.} = \frac{\sin \text{dec.}}{\sin \text{lat.}} \qquad \cos. \text{hour ang.} = \frac{\cot \text{lat.}}{\cot. \text{dec.}}$$

The first is for computing the latitude of the station when the declination of the star observed is known, or vice versâ. The other two are for the purpose of determining at what altitude to set the instrument, and at what time to look for the transit of any given star. In the latter two an approximate value of the latitude is to be used.

For the description of a large transit instrument, contrived for rapid reversal in the prime vertical, and having the telescope at one end of the axis, so that the striding level need never be removed from the supporting axis, see Struve's account of the Pulkova instrument, in 468 of the Astronomische Nachrichten of Schumacher. See also the Washington Astronomical Observations of 1845, introduction p. li. and p. 131. In these instruments there are 16 wires, 7 on each side of the middle one. The transits are taken over seven, and the instrument being quickly reversed, the transits are taken over the same seven in the reverse order.

The following is an example of one of Struve's observations. The 2d column reads upward.

January 15th, 1842, [illegible] Draconis.

Wires.	EAST VERTICAL. Telescope S. h m s	WEST VERTICAL. Telescope S. h m s
I.	17 54 30·75	19 42 51·4
II.	55 8·65	42 13·65
III.	55 44·4	41 38·0
IV.	56 22·25	40 59·85
V.	57 0·6	40 21·7
VI.	57 40·9	39 41·4
VII.	17 58 19·5	19 39 2·7
	Telescope N.	Telescope N.
VII.	18 1 4·0	19 36 17·85
VI.	1 45·5	35 37
V.	2 29·8	34 52·35
IV.	3 12·7	34 9·3
III.	3 57·6	33 24·7
II.	4 39·8	32 42·1
I.	18 5 26·35	19 31 55·6
	$i = 0''\cdot687$	$i' = 0'\cdot923$

Time of meridian transit not corrected for azimuth error of the instrument,

N.	18h	48m	41s·13
S.			41 ·05
Mean	18	48	41 ·09

A knowledge of the distance of each wire from the optical axis is unnecessary; for if this distance be denoted by c plus when the wire is N. of the optical axis, and when it is south, we have, t and t' being the corresponding hour angles, δ the declination, and ϕ the latitude,

$$-\sin c = \cos t \cos \delta \sin \phi - \sin \delta \cos \phi$$
$$+\sin c = \cos t' \cos \delta \sin \phi - \sin \delta \cos \phi$$
$$\therefore \quad o = (\cos t' + \cos t) \cos \delta \sin \phi - 2 \sin \delta \cos \phi$$
$$2 \sin c = (\cos t' - \cos t) \cos \delta \sin \phi$$

If we make $\frac{1}{2}(t' + t) = s$, and $\frac{1}{2}(t' - t) = u$, we have

$$\tan \delta = \tan \phi \cos s \cos u \qquad (1)$$
$$\sin c = \sin s \sin u \cos \delta \sin \phi$$

Formula (1) gives the declination, c being eliminated.

" (2) " distance of the wire from the optical axis.

The following is the application of (1) to the example above.

Wires.	I.	II	III.	IV.	V.	VI.	VII.	
	h m s	m s	m s	m s	m s	m s	m s	
W. — E. = 2t'	1 48 20·79	47 5 ·09	45 53·69	44 37·69	43 21·19	42 0·59	40 43·29	Telescope S.
= 2t	1 26 29·34	28 2·39	29 27·19	30 56·69	32 22·64	33 51·39	35 13·94	" N.
t')	48 42·53	48 46·87	48 50·22	48 53·60	48 55·96	48 58·05	48 59·31	
− t)	5 27·86	4 45·67	4 6 ·62	3 25·25	2 44·64	2 2·25	1 22·33	
	9·9901167	00871	00642	00411	00250	00107	00020	
	9·9998765	063	301	516	688	828	22	Mean.
φ log	10·2345728	2345728	2345728	2345728	2345728	2345728	2345728	
δ log. tan	10·2245660	662	671	655	666	663	670	·22456638
δ	59° 11′ 39″	39″·04	39″·23	38″·90	39″·12	39″·06	39″·21	59° 11′ 39″·071

If the inclination of the axis be denoted by i, which is the mean of the two inclinations, telescope N. and telescope S., then $\phi + i$ should be used in place of ϕ in the above formulas, or the correction for the declination should be

$$d\delta = \frac{\sin 2\delta}{\sin 2\phi} i$$

In the example above $\delta =$	59° 11′ 39″·071
Correction for inclination of axis $d\delta =$	0 ·814
Observed declination,	59° 11′ 39 ·885

The declination thus found is exact only on the hypothesis that the azimuth of the axis of rotation is zero. If there be an azimuth a we have $\pi = \frac{a}{\sin \phi}$ for the angle at the pole between the true meridian and the meridian of the instrument, or circle of declination perpendicular to the circle described by the optical axis of the instrument. The instant of transit of the star over the meridian of the instrument is the half sum of the times of corresponding transits E. and W. Thus for the star ϵ Draconis above, we have

Wires.	Telescope S.	Telescope N.
	h m s	h m s
I.	18 48 41·10	18 48 40·93
II.	41·15	41·25
III.	41·20	41·07
IV.	41·05	41
V.	41·15	41.15
VI.	41·15	40·95
VII.	41·10	40·97
Mean	18 48 41·13	41·05

Mean $18^h\ 48^m\ 41^s\cdot09$ time of meridian passage.

The instant of meridian passage p requires a small correction for the difference of inclinations of the axis in the two verticals E. and W. Denoting the former by i, and the latter by i'

$$\delta p = \frac{i - i'}{30\sqrt{\sin(\phi+\delta)\sin(\phi-\delta)}}\,\frac{\sin\delta}{\sin\phi}$$

In our example,

$$i = +0''\cdot687,\quad i' = +0''\cdot923$$
$$\therefore\quad i - i' = -0''\cdot236,\ \therefore\ dp = -0''\cdot08$$

and hence the true time of meridian passage by the instrument instead of $18^h\ 48^m\ 41^s\cdot09$, as above, is $p' = 18\ 48\ 41\cdot01$.

If α denote the right ascension of the star, and e the error of the clock, let $\alpha' = \alpha - e$ denote the time of passage of the star over the true meridian. Then, for the angle of the two meridians

$$\tau = p' - \alpha' \text{ in time}$$

and for the azimuth of the axis of rotation reckoned from the south round by the west,

$$a = 15\tau \sin\phi, \text{ in arc}$$

With this the correction of the observed declination for the axis of rotation becomes

$$d\delta = \tfrac{1}{4}(15\tau)^2 \sin 1'' \sin 2\delta$$

For stars near the zenith ϕ may be used instead of δ, and the formula becomes

$$d\delta = \tfrac{1}{4}(15\tau)^2 \sin 1'' \sin 2\phi$$

The clock error in the above example on 15th January, $18^h\ 48^m$ was $e =$ $8^s\cdot31$
The apparent right ascension of θ Draconis $\alpha =$ $18^h\ 48^m\ 50^s\cdot17$
$\therefore\ \alpha' =$ $18^h\ 48^m\ 41^s\cdot86$

$$\therefore\ \tau = -0^s\cdot85 \text{ in time, and } a = -11''\cdot0 \text{ in arc}$$

Finally the correction of the declination is

$$d\delta = 0''\cdot00017$$

too small to notice.

CONCLUDING NOTE.

author had designed giving the theory of eclipses and occultations, with their on to the determination of longitude, and also Bessel's method of measur- arc of the meridian; but as longitude by moon culminations is the node, and as measurements of the meridian are not at present being carried on or immediately contemplated in this country, it was thought expedient not to increase the size of a work intended for very general use. As, however, the method by occultations requires only a common telescope and a good timepiece, the reader is referred to the Appendix of the Nautical Almanac for 1836, pp. 134 and 145, for the necessary formulas, and an example of the computation of longitude by an occultation. (See also Lee's Tables and Formulas, p. 78, Part III.)